UNIQUELY ACHIEVABLE CENTRAL CANTOR SETS

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Let μ be a finite atomic measure that is both σ-finite and discrete. We may identify it with a measure on $\mathbb{N}$ satisfying

$$\mu(\{n\}) \geq \mu(\{n+1\}) \quad \text{for all } n.$$

Denoting $x_n := \mu(\{n\})$, we obtain

$$\operatorname{rng}(\mu) = \{\mu(E) : E \subset \mathbb{N}\} = \left\{ y \in \mathbb{R} : \exists A \subset \mathbb{N} \text{ such that } y = \sum_{n \in A} x_n \right\}.$$

Thus, the range of μ is the achievement set of a convergent series $\sum x_n$ with non-increasing, non-negative terms. The case when $x_n = 0$ for all but finitely many n is of little interest, so we additionally assume that $\sum x_n$ is a convergent series of positive, non-increasing terms. We say that a set $E \subset [0, +\infty)$ is *achievable* if $E = E(x_n)$ for some convergent series $\sum x_n$ of positive, non-increasing terms. We say that an achievement set $E(z_n)$ is *uniquely achievable* (or *uniquely determined*) if there is no other sequence (y_n) such that $E(z_n) = E(y_n)$. The notion of the range of a finite atomic measure naturally leads to the following two questions:

- For which sets $E \subset [0, +\infty)$ does there exist a measure μ such that $E = \operatorname{rng}(\mu)$?
- For which sets $E \subset [0, +\infty)$ does there exist exactly one measure μ such that $E = \operatorname{rng}(\mu)$?

The first question, raised by Kakeya in 1914 in the context of achievement sets [6], [7] and by Rényi in 1970 in the context of purely atomic measures [11], received a partial topological answer in the celebrated Guthrie–Nymann classification [5]. However, we still have very few tools for determining whether a given set is achievable. The second question, concerning the possibility of recovering a purely atomic measure from its range, remains open in general. The case where E is an interval or a finite union of intervals has been studied in [8], [9], [10], and [3]. Almost nothing is known when E is a Cantorval, apart from the nontrivial observation that the Guthrie–Nymann Cantorval $\frac{1}{4}E(3, 2; \frac{1}{4})$ is uniquely achievable [3, Thm. 5.2]. Finally, the case where E is a Cantor set was investigated in [3], where several interesting partial results were obtained. During my talk, I will present the following complete characterization for the case in which E is a central Cantor set.

Theorem. *Let E be a central Cantor set with $\min E = 0$, and let (x_n) be its unique fast convergent representation. Then E is uniquely achievable if and only if $x_n \neq 2x_{n+1}$ for all n.*

The proof relies heavily on the theory of minimal centers of distances of central Cantor sets, initiated in [4], further developed in [2], and completed in [1].

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