Scaling limit for small blocks in the Chinese restaurant process

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Let $\mathcal{P}_n,\ n \in \mathbb{N}$, denote the partition of $[n] = \{1, \ldots, n\}$ formed at the nth step of the Chinese restaurant process CRP(θ), where $\theta > 0$ is a real parameter (see, e.g., [1]). For convenience, we list the elements within each block of $\mathcal{P}_n$ in ascending order. The growth of each block is described by the following scheme:

$$\{k_1\} \longrightarrow \{k_1, k_2\} \longrightarrow \ldots \longrightarrow \{k_1, ..., k_N\} \longrightarrow \{k_1, ..., k_N, k_{N+1}\} \longrightarrow \ldots \tag{1}$$

Let $A(k_1, \ldots, k_N, k_{N+1}) = \{\{k_1, \ldots, k_N, k_{N+1}\} \in \mathcal{P}_{k_{N+1}}\}$ be the random event indicating the existence of a block that, up to the step k_{N+1}, evolves as described by (1). For a fixed $N \in \mathbb{N}$, the structure and dynamics of all blocks up to the times they reach size $N + 1$ are uniquely determined by the infinite collection of events $\{A(k_1, \ldots, k_N, m) : 1 \le k_1 < \ldots < k_N < m\}$.

Main result. Let $\mathbb{X}_N = \{(x_1, \ldots, x_N, y) \in [0, +\infty)^{N+1} :\ x_1 \le \ldots \le x_N \le y\}$. Considering it as a measurable space, we equip it with the Borel σ-algebra and a special localizing ring of bounded sets. Define a sequence of random point measures $\Xi_N^{(n)}$, $N \in \mathbb{N}$, by

$$\Xi_n^{(N)} = \sum_{1 \le k_1 < \ldots < k_N < m} \delta_{\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}, \frac{m}{n}\right)} \mathbb{1}_{A(k_1, \ldots, k_N, m)}.$$

Theorem. $\Xi_n^{(N)}$ *vaguely converges in distribution as* $n \to \infty$ *to the Poisson random measure* $\Xi^{(N)}$ *on* $\mathbb{X}_N$ *with intensity measure* μ *defined by* $\mathrm{d}\mu^{(N)} = \theta \frac{N!}{y^{N+1}} \mathrm{d}x_1 \ldots \mathrm{d}x_N \mathrm{d}y$.

Applications. The theorem above, combined with the continuous mapping theorem, allows us to derive limit results for a variety of CRP characteristics with limits given in an explicit form. We present a few examples of both classical and functional limit theorems for the following characteristics:

- the joint distribution of block counts at times n and $\lfloor \alpha n \rfloor$, $\alpha > 1$;
- the number of singletons in $\mathcal{P}_{\lfloor nt \rfloor}$;
- the first singleton in $\mathcal{P}_{\lfloor nt \rfloor}$;
- the dynamics of singletons with a short lifetime.

The talk is based on the joint work with Andrii Ilienko [2]. References [1] Pitman Jim. Combinatorial stochastic processes. Ecole d'Et ?e de Probabilit ?es de Saint-Flour XXXII – 2002. — Berlin: Springer, 2006. — Vol. 1875 of Lect. Notes Math. [2] O. Galganov, A. Ilienko. Scaling limit for small blocks in the Chinese restaurant process. — 2025. — Submitted. DOI: https://doi.org/10.48550/arXiv.2502.05578.

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