Switching problem on a graph

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A. Skorokhod in 1961 formulated the reflection problem in his work [1], which subsequently proved to be an effective method for constructing reflected diffusions. Based on this reflection problem, a mapping was built that assigns to a function the solution of Skorokhod's reflection problem for that function. This mapping turned out to be useful for solving various problems in queueing theory (see, for example, [2]). Skorokhod's reflection problem has been generalized by different authors. In particular, A. Pilipenko in the article [3] posed the problem of jump-like reflection, and this work made it possible to construct Brownian motion with jump-like reflection (see [4]).

The aim of our work is to construct a similar mapping for natural gluing of functions on a graph. We investigate the properties of this mapping and apply it to the study of random walks, using the continuous mapping Theorem and Donsker's invariance principle.

Definition 1. Let $m \in \mathbb{N}$. We say that the function $\vec{X}$ is a solution to the *graph switching problem* for function $\vec{W}$ and compensating function $\vec{H}$ if for each $i \in \overline{1,m}$ the function $X_i(t)$ from the vector $\vec{X}(t) = (X_1(t), X_2(t), ..., X_m(t))$ can be represented as

$$X_i(t) = W_i(T_i(t)) + H_i(\widetilde{T}(t))$$

and there is a sequence $\{a_k, b_k \mid k \geq 0\} \subset \mathbb{R}_+$ such that this sequence and functions $T_i \colon \mathbb{R}_+ \mapsto \mathbb{R}_+$, $\widetilde{T} \colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfy the following conditions:

1. $0 = a_0 \leq b_0 \leq a_1 \leq b_1 \leq ... \leq a_n \leq b_n \leq ... \leq +\infty;\ a_k \to +\infty$.
2. $\widetilde{T}$ is a non-decreasing right-continuous function and $\widetilde{T}(0) = 0$.
3. For all $i \in \overline{1,m}$ the function T_i is a piecewise linear function that has derivative at each point of $\mathbb{R}_+$ except maybe $\{a_k, b_k \mid k \geq 0\}$.
4. For all $t \in \mathbb{R}_+ \setminus \{a_k, b_k \mid k \geq 0\}$ and for all $i \in \overline{1,m}$ we have $T_i'(t) \in \{0, 1\}$ and $T_i(0) = 0$.
5. For all $t \in A = \cup_{k=0}^{\infty}(a_k, b_k)$ holds $\sum_{i=1}^{m} T_i'(t) = 1$.
6. $\int_0^\infty \mathbb{I}_A(t)\, \mathrm{d}\widetilde{T}(t) = 0$.
7. For $B = \cup_{k=0}^{\infty}(b_k, a_{k+1})$ for all $i \in \overline{1,m}$ holds $\int_0^\infty \mathbb{I}_B(t)\, \mathrm{d}T_i(t) = 0$.

We shall denote a solution $\vec{X}$ of the *graph switch problem* for functions $\vec{W}$, $\vec{H}$ in the following way: $\vec{X} \in \mathcal{R}(\vec{W}, \vec{H})$.

Theorem 1. *For $n \in \mathbb{N}$ let $\vec{X}^{(n)}$ be a function from $\mathcal{R}(\vec{W}^{(n)}, \vec{H}^{(n)})$ and let the following conditions hold:*

1. *The sequence of functions $\{\vec{W}^{(n)}\}_{n\geq 1}$ converges locally uniformly to the function $\vec{W}$ from $C(\mathbb{R}_+, \mathbb{R}^m)$ on $\mathbb{R}_+$ as $n \to \infty$.*
2. *The sequence of functions $\{\vec{H}^{(n)}\}_{n\geq 1}$ converges locally uniformly to the function $\vec{H}$ from $C(\mathbb{R}_+, \mathbb{R}^m)$ on $\mathbb{R}_+$ as $n \to \infty$.*

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3. *For all $T>0$ and $\forall i\in\overline{1,m}$ we have* $\liminf\limits_{n\to\infty}\inf_{t\in[0,T]} X_i^{(n)}(t)\ge 0$.

4. *For all $T>0$ and $\forall i\in\overline{1,m}$ we have* $\limsup\limits_{n\to\infty}\sup\limits_{t\in[0,T]\cap\overline{B^{(n)}}} X_i^{(n)}(t)\le 0$.

5. *The functions $H^{(n)}(t)=\sum_{i=1}^m H_i^{(n)}(t)$ converge locally uniformly to the function t on $\mathbb{R}_+$ as $n\to\infty$.*

6. *The sequence of functions $T^{(n)}(t):=\sum_{i=1}^m T_i^{(n)}(t)$ converges locally uniformly to the function t on $\mathbb{R}_+$ as $n\to\infty$.*

Then there exists a subsequence n_k and functions $T_i\colon\mathbb{R}_+\mapsto\mathbb{R}$, $i\in\overline{1,m}$ and $\widetilde{T}\colon\mathbb{R}_+\mapsto\mathbb{R}$ such that for all $i\in\overline{1,m}$ the functions $X_i^{(n_k)}$ converge locally uniformly to the function $X_i(\cdot):=W_i(T_i(\cdot))+H_i(\widetilde{T}(\cdot))$ as $k\to\infty$ and for all $i\in\overline{1,m}$ the functions $T_i^{(n_k)}$ converge to T_i and the functions $\widetilde{T}^{(n_k)}$ converge to $\widetilde{T}$ locally uniformly. Moreover,

$$\begin{cases}\sum\limits_{i=1}^m T_i(t)=t,\\ l_i(T_i)=H_i(\widetilde{T}(t)),\quad i=\overline{1,m}.\end{cases}$$

The talk is based on the joint work with Andrey Pilipenko.

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