Cantorvals: emergence, structure, open problems

Fractal analysis seminar

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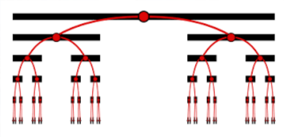
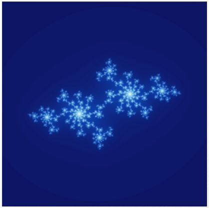
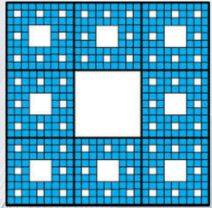
What is fractal set?

A set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension is called fractal.

(B. Mandelbrot, 1977)

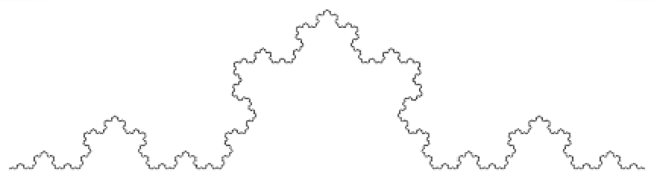


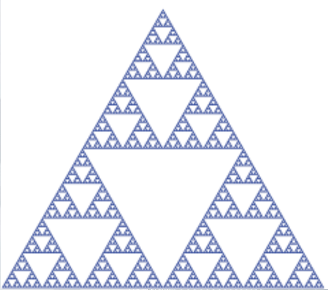


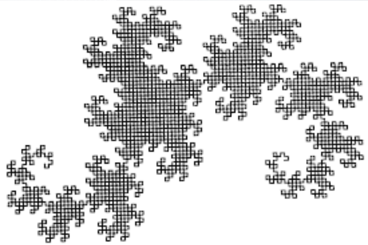


Fractals with empty interior.

Preface III

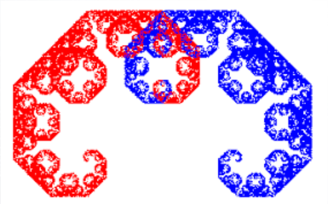


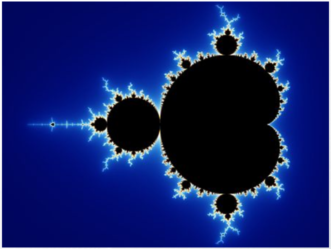


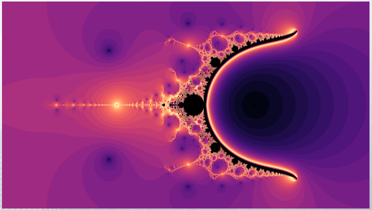


Fractals with empty interior.

Preface IV







Fractals with non-empty interior.

Preface V

Cantorval – a perfect set on the real line with non-empty interior and fractal boundary.

Natural emergence of Cantorval I

The term "Cantorval" was first coined by P. Mendes and F. Oliveira, who studied arithmetic sums

$$C_1 \oplus C_2 = \{x_1 + x_2 : x_1 \in C_1, x_2 \in C_2\}$$

of two homogeneous Cantor sets C_1 and C_2 with zero Lebesgue measure. As a result of such a sum, there occurs a perfect set combining the properties of intervals and a nowhere dense set simultaneously.

Natural emergence of Cantorval II

Anisca R., and M. Ilie (2023): On the structure of arithmetic sums of Cantor sets associated with series. - Results Math. 78:5, article no. 5.

Pourbarat, M. (2022): Topological structure of the sum of two homogeneous Cantor sets. Ergodic Theory and Dynamical Systems, 43:5, 1712-1736.

Filipczak, T., and P. Nowakowski (2023): Conditions for the difference set of a central Cantor set to be a Cantorval. - Results Math. 78, art. no. 166.

Mendes, P., and F. Oliveira (1994): On the topological structure of the arithmetic sum of two cantor sets. - Nonlinearity. 7:2, 329–343.

Natural emergence of Cantorval III

Cantorvals are also one of the three possible topological types of the set of subsums for a convergent positive series $\sum a_n$, i.e., the set

$$E(a_n) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n a_n : \quad (\varepsilon_n) \in \{0,1\}^{\mathbb{N}} \right\}.$$

Nymann, J., and R. Sáenz (2000): On a paper of Guthrie and Nymann on subsums of infinite series. - Colloq. Math. 83:1, 1–4.

Natural emergence of Cantorval IV

A Cantorval can arise as the attractor of an iterated function system (IFS) with overlaps, or even without overlaps in certain special cases. Iterated function systems provide a relatively simple framework for modeling self-similar sets in general, and fractal sets in particular. IFSs without overlaps, which satisfy the open set condition, have been well studied by B. Mandelbrot, J. Hutchinson, and K. Falconer. In contrast, IFSs with overlaps remain largely understudied due to the considerable difficulties they present.

Natural emergence of Cantorval V

Fraser, J., Henderson, A., Olson, E., Robinson, J. (2015): On the Assouad dimension of self-similar sets with overlaps. Advances in Mathematics, 273, 188-214.

Banakh, T., Bartoszewicz, A., Filipczak, M., Szymonik, E. (2015): Topological and measure properties of some self-similar sets. Topol. Methods Nonlinear Anal. 46:2, 1013-1028.

Banakh T., Bartoszewicz A., Glab S., Szymonik E. (2012): Algebraic and topological properties of some sets in ℓ_1 // Colloq. Math. 129, 75–85.

Banakiewicz, M. (2019): The Lebesgue measure of some M-Cantorval. - Journal of Mathematical Analysis and Applications. 471, 170–179.

Natural emergence of Cantorval VI

Cantorvals are also connected to the study of the spectrum of a discrete two-dimensional Schrödinger operator with separable aperiodic potential given by the Fibonacci sequence in both directions in an intermediate coupling regime.

D. Damanik, M. Embree and A. Gorodetski (2015): Spectral properties of Schrödinger operators arising in the study of quasicrystals, in Mathematics of Aperiodic Order, eds. J. Kellendonk, D. Lenz and J. Savinien, Birkhauser, Basel, pp. 307–370.

D. Damanik, A. Gorodetski and B. Solomyak (2015): Absolutely continuous convolutions of singular measures and an application to the square Fibonacci Hamiltonian, Duke Math. J. 164 (2015) 1603–1640.

Natural emergence of Cantorval VII

The existence of a large family of Cantorvals is observed in the projection description of primitive two-letter substitutions. These arise from the study of geometric, self-similar realizations of aperiodic sequences (i.e., one-dimensional quasicrystals) with two symbols, which can be described as regular model sets.

Baake, M., Gorodetski, A., Mazac, J. (2024): A naturally appearing family of Cantorvals. Lett. Math. Phys., 114, article no. 101.

Natural emergence of Cantorval VIII

Cantorvals naturally appear in various branches of mathematics, including mathematical analysis, dynamical systems, number theory, and probability theory. Despite the relatively recent emergence of the topic, it remains largely understudied. In this context, the interest in Cantorvals is considerable and continues to grow.

The set of subsums of series I

Let us consider a convergent positive series

$$\sum_{n=1}^{\infty} a_n = r < \infty, a_n > 0.$$

We also introduce the notation

$$r_n := \sum_{i=n+1}^{\infty} a_i,$$

where r_n is often called the n-th **tail** or **remainder** of the series.

The set of subsums of series II

By $E(a_n)$ we denote **the set of** all **subsums** for the series $\sum a_n$, i.e.,

$$E(a_n) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n a_n : \ (\varepsilon_n) \in \{0,1\}^{\mathbb{N}} \right\}$$

Some researchers also refer to this object as as the **achievement set** of the sequence (a_n). For simplicity, we say that a set A is achievable if it coincides with the set of subsums of some series. The study of the set of subsums of numerical series was initiated by S. Kakeya in 1914. It is easy to see that $E(a_n) \subset [0, r]$, where $r = \sum_{n=1}^{\infty} a_n$. Moreover, $E(a_n)$ is a perfect, symmetric set. The object was rediscovered by H. Hornich (1941) and K. Menon (1948).

The set of subsums of series III

Theorem (Kakeya-Hornich-Menon, 1914-1948)

Let $\sum_{n\geq 1} a_n$ be a convergent positive series with non-increasing terms, i.e., $a_n \geq a_{n+1}$ for any $n \in \mathbb{N}$. Then $E(a_n)$ is:

1. *a finite union of closed bounded intervals if and only if $a_n \leq r_n$ for all but finitely many $n \in \mathbb{N}$;*
2. *a closed interval if and only if $a_n \leq r_n$ for all $n \in \mathbb{N}$;*
3. *homeomorphic to the Cantor set if $a_n > r_n$ for all but finitely many $n \in \mathbb{N}$.*

For simplicity, we say that $\sum a_n$ satisfies the Kakeya condition if either $a_n \leq r_n$ or $a_n > r_n$ holds for all but finitely many values of n. A more intricate case arises when the inequalities are mixed – that is, when $a_i > r_i, i \in A^-$ and $a_j \leq r_j, j \in A^+$, where A^- and A^+ are countable sets such that $A^+ \cup A^- = \mathbb{N}$.

The set of subsums of series IV

Kakeya conjectured that $E(a_n)$ is either a Cantor set or a finite union of closed intervals. This assumption was refuted first by A. Vainshtein and B. Shapiro (1980), and later by C. Ference (1984). One of the most familiar examples was described by J. Guthrie and J. Nymann who showed that the set of subsums for the series

$$\frac{3}{4}+\frac{2}{4}+\frac{3}{4^2}+\frac{2}{4^2}+\frac{3}{4^3}+\frac{2}{4^3}+\cdots+\frac{3}{4^i}+\frac{2}{4^i}+\ldots,$$

is neither a finite union of closed intervals nor a Cantor set, demonstrating the existence of a third type of set.

Guthrie J. A., Nymann J. E. The topological structure of the set of subsums of an infinite series // Colloq. Math. – 1988. – Vol. 55, № 2. – P. 323-327.

The set of subsums of series V

Theorem (Guthrie-Nymann-Saenz, 2000)

The set of subsums for a convergent positive series is one of the following three types:

1. *a finite union of closed intervals;*
2. *homeomorphic to the classic Cantor set (Cantor set);*
3. *an M-Cantorval, or a set homeomorphic to*

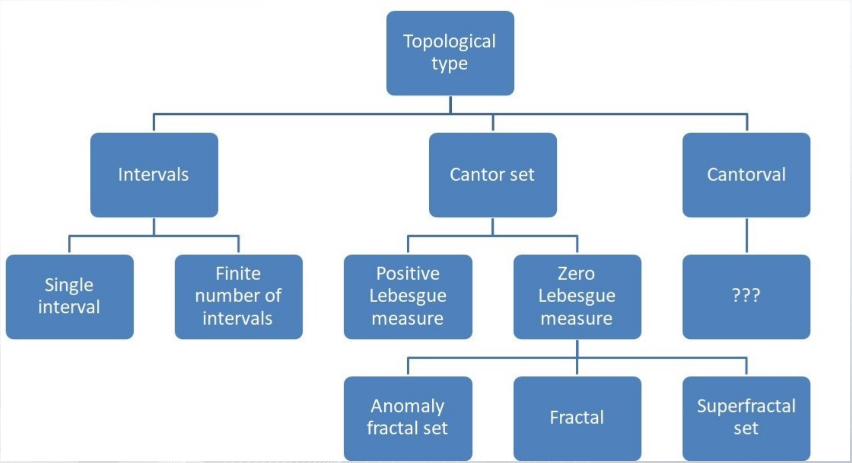
$$Y \equiv C \cup \bigcup_{n=1}^{\infty} G_{2n-1},$$

where C is the Cantor ternary set, G_n is the union of the 2^{n-1} open middle thirds which are removed from $[0,1]$ at the n-th step in the construction of C.

$G_1 = \left(\frac{1}{3}, \frac{2}{3}\right), \quad G_2 = \left(\frac{2}{9}, \frac{3}{9}\right) \bigcup \left(\frac{7}{9}, \frac{8}{9}\right),$

$G_3 = \left(\frac{1}{27}, \frac{2}{27}\right) \bigcup \left(\frac{7}{27}, \frac{8}{27}\right) \bigcup \left(\frac{19}{27}, \frac{20}{27}\right) \bigcup \left(\frac{25}{27}, \frac{26}{27}\right)$, *and so on.*

The set of subsums of series VI



The set of subsums of series VII

Examples:

1. $\sum_{n=1}^{\infty} \frac{1}{2^n} \to E \equiv [0,1];$
2. $\sum_{n=1}^{\infty} a_n = 2 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^i} + \cdots \to E \equiv [0,1] \cup [2,3];$
3. $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{10^n} \right) \to E$ is a fat Cantor set;
4. $\sum_{n=1}^{\infty} \frac{1}{n!} \to \dim_H E = 0;$
5. $\sum_{n=1}^{\infty} q^n, 0 < q < \frac{1}{2} \to E$ is a fractal with $\dim_H E = -\log_q 2;$
6. GN-series $\to E$ is a Cantorval;

Multigeometric series I

The necessary and sufficient conditions for the set of subsums to be a Cantorval or a Cantor set still remain unknown. Despite significant progress for certain classes of series, the problem remains quite difficult in the general setting. In this context, researchers focus on series whose terms belong to sequences that satisfy certain homogeneity conditions (for example – sequences defined by a finite number of parameters, an explicit formula for the general term, or a recurrence relation.).

Multigeometric series II

The most significant results in that direction were obtained for multigeometric series

$$k_1 + k_2 + \dots + k_m + k_1 q + \dots + k_m q + \dots + k_1 q^i + \dots + k_m q^i + \dots,$$

where $k_1, k_2, \dots, k_m$ are fixed positive scalars, $q \in (0, 1)$.

It is easy to see that such a set is the attractor of a homogeneous IFS with contraction ratio q and translations formed by all possible combinations of $k_1, k_2, \dots, k_m$.

Multigeometric series III

Certain conditions ensuring that $E(a_n)$ is either a Cantorval or a Cantor set, as well as solutions to related problems concerning such Cantorvals, were established in the following works:

Jones, R. (2011): Achievement Sets of Sequences. The American Mathematical Monthly, 118:6, 508–521.

Bartoszewicz, A., M. Filipczak, and E. Szymonik (2014): Multigeometric sequences and Cantorvals. - Central European Journal of Mathematics. 12:7, 1000–1007.

Ferdinands, J., and T. Ferdinands (2019): A family of Cantorvals. - Open Mathematics. 17, 1468–1475.

Multigeometric series IV

Banakiewicz, M., and F. Prus-Wisniowski (2017): M-Cantorvals of Ferens type. Mathematica Slovaca., 67:4, 907-918.

Glab, S., Marchwicki, J. (2023): Set of Uniqueness for Cantorvals. Results Math. 78, article no. 9.

Banakiewicz M. (2019): The Lebesgue measure of some M-Cantorval // Journal of Mathematical Analysis and Applications. – 2019. – 471, № 1-2. – P. 170–179.

Bielas, W., Plewik, S., Walczynska, M. (2018): On the center of distances. European Journal of Mathematics, 4, 687-698.

Multigeometric series V

In contrast, the following articles discuss non-multigeometric series whose sets of subsums are Cantorvals:

Pratsiovytyi, M., and D. Karvatskyi (2023): Cantorvals as sets of subsums for a series connected with trigonometric functions. - Proceedings of the International Geometry Center. 15:3-4, 262–271.

Vinishin, Y., V. Markitan, M. Pratsiovytyi, and I. Savchenko (2019): Positive series having Cantorvals as sets of subsums. - Proceedings of the International Geometry Center. 12:2, 26–42.

Some results of IFS theory I

Let us consider a finite collection of contractive maps on $\mathbb{R}$:

$$\Phi = \{f_i(x) = \lambda_i \cdot x + t_i\}_{i=1}^{N}$$

where $0 < \lambda_i < 1$ and $t_i \in \mathbb{R}$ that is commonly referred to as **iterated function system (IFS)**. For every IFS, there exists a unique non-empty compact set $K \subseteq \mathbb{R}$ such that

$$K = \bigcup_{i=1}^{N} f_i(K),$$

which is commonly called the **attractor** of the IFS Φ. Furthermore, K is the closure of the set of all fixed points of finite compositions $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_p}$.

Some results of IFS theory II

In the case when all contractions are similarities, Φ generates a self-similar set. If, in addition, $\lambda_i = \lambda$ for all $i \in \{1, \dots, N\}$, then Φ is called **homogeneous** or **equicontractive**.

A positive number α such that

$$\lambda_1^\alpha + \lambda_2^\alpha + \cdots + \lambda_N^\alpha = 1$$

is called the **similarity dimension** of Φ.

It is well known that

$$\dim_H K \leq \min\{1, \alpha\}.$$

Some results of IFS theory III

An IFS $\Phi = \{f_1, f_2, \dots, f_N\}$ is said to satisfy the Open Set Condition (OSC) if there exists a non-empty bounded open set V such that:

1. $f_i(V) \subseteq V$ for all i;
2. $f_i(V) \cap f_j(V) = \emptyset$ for all $i \neq j$.

Due to Hutchinson, if an IFS satisfies the OSC then

$$0 < H^\alpha(K) < \infty$$

and consequently

$$\dim_H(K) = \alpha.$$

Some results of IFS theory IV

Theorem (Bandt-Graf-Schief, 1992-1994)

The following conditions are equivalent:

- Strong open set condition (SOSC);
- Open set condition (OSC);
- $H^\alpha(K) > 0$, where α is the similarity dimension of IFS.

Theorem C (A.Schief, 1994)

Let the similarity dimension of an IFS be equal to $\alpha = 1$. Then K contains interior points if and only if $\lambda(K) > 0$. Moreover, in this case, the IFS satisfies the OSC.

On topological structure of Cantorvals I

We sketch some basic topological properties of an arbitrary bounded perfect set A. Connected components of $A \subset \mathbb{R}$ are either closed intervals or singletons. Intervals that are connected components of the set A will be called A-*intervals*, while one-point connectivity components of A will be called *loose points of* A. Bounded open intervals that are connected components of the complement $A^c = R \setminus A$ will be called A-*gaps*.

How can we describe the interior and boundary of an achievable set?

On topological structure of Cantorvals III

Cantorval can be formally defined as a nonempty compact real subspace which is the closure of its interior and the endpoints of any nontrivial component of this set are accumulation points of trivial components.

The structure of some Cantorvals I

In 1988, John Guthrie and John Nymann considered the set

$$X = \left\{ \sum_{n=1}^{\infty} \frac{\alpha_n}{4^n} : (\alpha_n) \in \{0, 2, 3, 5\}^{\mathbb{N}} \right\},$$

of subsums for the series

$$\frac{3}{4} + \frac{2}{4} + \frac{3}{4^2} + \frac{2}{4^2} + \frac{3}{4^3} + \frac{2}{4^3} + \cdots + \frac{3}{4^i} + \frac{2}{4^i} + \ldots,$$

which will further be called *Guthrie-Nymann's series*. This set contains the interval $[3/4, 1]$, but it is not a finite union of closed intervals. It exemplifies a third possible type for $E(a_n)$ – a mixed type.

The structure of some Cantorvals II

It is easy to observe that X simultaneously possesses the properties of being achievable and self-similar, as it is the attractor of the IFS given by:

1. $w_1(x) = \dfrac{x}{4}$;
2. $w_2(x) = \dfrac{x}{4} + \dfrac{2}{4}$;
3. $w_3(x) = \dfrac{x}{4} + \dfrac{3}{4}$;
4. $w_4(x) = \dfrac{x}{4} + \dfrac{5}{4}$.

In this context, the set X is significantly more tractable for further study.

The structure of some Cantorvals III

Let X_I denote the interior and X_C denote the boundary of X. The set X has the following properties:

Property 1

The interval $\left[\frac{2}{3}, 1\right]$ is contained in the Cantorval X.

For any $x \in \left[\frac{3}{4}, 1\right]$ there exists $(\alpha_n) \in \{0, 1, 2, 3\}^{\mathbb{N}}$ with $\alpha_1 = 3$ such that

$$x = \Delta^4_{\alpha_1 \ldots \alpha_n \ldots} = \sum_{n=1}^{\infty} \frac{\alpha_n}{4^n}.$$

We can prove that for any $(\alpha_n) \in \{0, 1, 2, 3\}^{\mathbb{N}}$ there exists $(\beta_n) \in \{0, 2, 3, 5\}^{\mathbb{N}}$ such that

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{4^n} = \sum_{n=1}^{\infty} \frac{\beta_n}{4^n}.$$

Property 2

The set $X \setminus \left(\frac{2}{3}, 1\right)$ is a union of pairwise disjoint affine copies of X. In particular, this union contains two isometric copies of $\frac{1}{4^n} \cdot X$ for each $n \in \mathbb{N}$.

$H = \left[0, \frac{5}{3}\right]$ – convex hull of X;
$X = \bigcup_{i=1}^{4} w_i(X)$ as self-similar set;
$w_1(H) \cap w_i(H) = \emptyset,\ \text{for } 2 \leq i \leq 4;$
$w_i(H) \cap w_4(H) = \emptyset,\ \text{for } 1 \leq i \leq 3;$
$w_1(H) \cap \left[\frac{2}{3}, 1\right] = \emptyset;$
$w_4(H) \cap \left[\frac{2}{3}, 1\right] = \emptyset.$
$\emptyset \neq w_2(H) \cap w_3(H) \subset \left[\frac{2}{3}, 1\right];$

The structure of some Cantorvals V

$$w_2(X) = \bigcup_{i=1}^{4} w_2 \circ w_i(X);$$
$$w_2 \circ w_1(H) \cap w_2 \circ w_i(H) = \emptyset, \quad \text{for} \quad 2 \le i \le 4;$$
$$w_2 \circ w_1(H) \cap \left[\tfrac{2}{3}, 1\right] = \emptyset;$$
$$w_2 \circ w_i(H) \subset \left[\tfrac{2}{3}, 1\right] \quad \text{for} \quad 3 \le i \le 4;$$

$$\underbrace{w_2 \circ \cdots \circ w_2}_{n-1} \circ w_1(H) \cap \underbrace{w_2 \circ \cdots \circ w_2}_{n-1} \circ w_i(H) = \emptyset, \quad \text{for} \quad 2 \le i \le 4;$$
$$\underbrace{w_2 \circ \cdots \circ w_2}_{n-1} \circ w_1(H) \cap \left[\tfrac{2}{3}, 1\right] = \emptyset;$$
$$\underbrace{w_2 \circ \cdots \circ w_2}_{n-1} \circ w_i(H) \subset \left[\tfrac{2}{3}, 1\right] \quad \text{for} \quad 3 \le i \le 4;$$
$$\lim_{n\to\infty} w_2^{n-1} \circ w_1(H) = \tfrac{2}{3}$$

The structure of some Cantorvals VI

Property 3

The set X_C is a union of pairwise disjoint affine copies of itself with similarity ratios $1/4^n$, $n \in \mathbb{N}$, namely

$$X_C = \bigsqcup_{n\in\mathbb{N}} \left(\bar{C}_n^l \sqcup \bar{C}_n^r\right),$$

where

$$\bar{C}_n^l = \underbrace{w_2 \circ \cdots \circ w_2}_{n-1} \circ w_1(X_C) = \sum_{i=1}^{n-1} \frac{2}{4^i} + \frac{1}{4^n} \cdot X_C,$$

$$\bar{C}_n^r = \underbrace{w_3 \circ \cdots \circ w_3}_{n-1} \circ w_4(X_C) = h\left[\bar{C}_n^l\right]$$

represent the left and right copies.

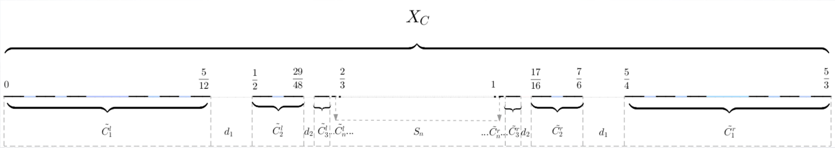


Рис.: The set X_C consists of countable disjoint affine copies $\bar{C}_n^l, \bar{C}_n^r$.

Such sets are referred to as **countable self-similar** or N**-self-similar**, and were studied in detail by M. Pratsiovytyi, H. Fernau, M. Moran, and D. Mauldin. Since all the copies of N-self-similar set X_C are disjoint, its Hausdorff dimension coincides with N-similarity dimension (analogue of similarity dimension). This dimension can be determined as the unique solution to the following equation:

$$2\left(\frac{1}{4}\right)^x + 2\left(\frac{1}{4^2}\right)^x + 2\left(\frac{1}{4^3}\right)^x + \cdots + 2\left(\frac{1}{4^n}\right)^x + \cdots = 1,$$

that implies

$$\frac{4^{-x}}{1-4^{-x}} = \frac{1}{2} \quad \Rightarrow \quad 4^x = 3 \quad \Rightarrow \quad x = \log_4 3.$$

The structure of some Cantorvals IX

Property 4

The distance between adjacent affine copies $\bar{C}_n^l, \bar{C}_{n+1}^l$ and $\bar{C}_n^r, \bar{C}_{n+1}^r$ can be computed by the formula

$$d_n = d\left(\bar{C}_n^l, \bar{C}_{n+1}^l\right) = d\left(\bar{C}_n^r, \bar{C}_{n+1}^r\right) = \frac{1}{3} \cdot \frac{1}{4^n}.$$

Property 5

The distance between the symmetric affine copies $\bar{C}_n^l$ and $\bar{C}_n^r$ can be calculated as

$$s_n = d\left(\bar{C}_n^l, \bar{C}_n^r\right) = \frac{5}{3} - 2 \cdot \left(\sum_{i=1}^{n-1} \frac{2}{4^i} + \frac{1}{4^n} \cdot \frac{5}{3}\right).$$

The structure of some Cantorvals X

Theorem

The Cantorval X, which is the set of subsums of the Guthrie-Nymann series, can be represented as $X = X_I \bigsqcup X_C$, where X_I is a countable union of open intervals with total Lebesgue measure equal to 1, X_C is a Cantor set of zero Lebesgue measure and Hausdorff dimension $\dim_H X_C = \log_4 3$.

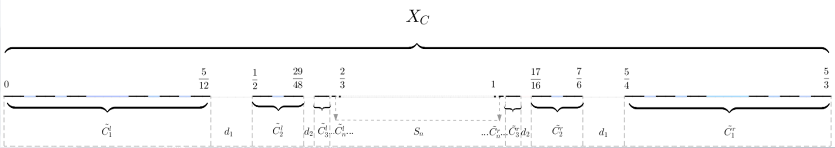


Рис.: The set X_C consists of countable disjoint affine copies $\bar{C}_n^l, \bar{C}_n^r$.

The structure of some Cantorvals XI

The above result can be extended to a countable family of Cantorvals of the form

$$X(m) = \left\{ \sum_{n=1}^{\infty} \frac{\alpha_n}{(2m+2)^n} : (\alpha_n) \in \{0, 2, 3, \ldots, 2m+1, 2m+3\}^{\mathbb{N}} \right\},$$

which arise as the sets of subsums of the series

$$3q + \underbrace{2q + \cdots + 2q}_{m} + 3q^2 + \underbrace{2q^2 + \cdots + 2q^2}_{m} + \ldots$$

$$\cdots + 3q^n + \underbrace{2q^n + \cdots + 2q^n}_{m} + \ldots$$

where $q = 1/(2m+2), m \in \mathbb{N}$.

The structure of some Cantorvals XII

Theorem

The Cantorval $X(m)$ can be represented as the disjoint union $X = X_I(m) \bigsqcup X_C(m)$, where $X_I(m)$ is a countable union of open intervals with total Lebesgue measure equal to 1, $X_C(m)$ is a Cantor set with zero Lebesgue measure and Hausdorff dimension $\dim_H X_C = \log_{2m+2} 3$.

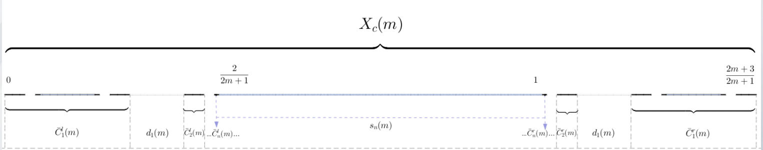
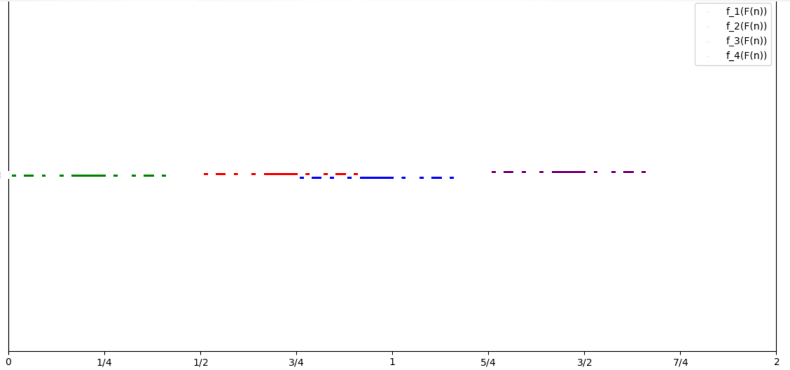


Рис.: The set $X_C(m)$ consists of countable disjoint affine copies $\bar{C}^l_n(m), \bar{C}^r_n(m)$.

Moreover, all the described Cantorvals satisfy the Open Set Condition. For the Guthrie-Nymann set, we have



We consider a class of positive functions f defined on the interval and study the topological behavior, depending on x, of the set of all possible subsums for the series

$$\sum_{n=1}^{\infty} w_n(x) = k_1 f(x) + \dots + k_m f(x) + k_1 f(x^2) + \dots + k_m f(x^2) + \dots$$

$$\dots + k_1 f(x^n) + \dots + k_m f(x^n) + \dots,$$

where $k_1 \geq k_2 \geq \dots \geq k_m$ are fixed positive scalars, f satisfies certain special conditions. We can write down the following

$$E\big(w_n(x)\big) = \left\{ \sum_{n=1}^{\infty} \alpha_n f(x^n),\, \alpha_n \in A \right\},\ A = \left\{ \sum_{i=1}^{m} c_i k_i,\, c_i \in \{0, 1\} \right\}.$$

Special Condition.
A function f is *locally increasing and power bounded* (at 0) if there exist $\gamma \in (0,1)$, and $a, b, t \in \mathbb{R}^+$ such that f is monotone increasing in $[0,\gamma]$ and

$$a \cdot x^t \leq f(x) \leq b \cdot x^t$$

for every $x \in [0,\gamma]$. We denote by M the class of locally increasing at 0 and power bounded functions.

Generalised multigeometric series III

Remark

If $f \in C^{t+1}([0,1))$ such that $f^{(i)}(0) = 0$ for $0 \le i < t$ and $f^{(t)}(0) > 0$, then $f \in M$.

In this case, a simple estimate can be obtained using the Taylor (Maclaurin) expansion:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

and it is possible to define

$$a := \frac{1}{t!}\min\left\{f^{(t)}(\zeta) : \zeta \in [0,\gamma]\right\}, \quad b := \frac{1}{t!}\max\left\{f^{(t)}(\zeta) : \zeta \in [0,\gamma]\right\}.$$

Generalised multigeometric series IV

Let us define

$$K = \sum_{i=1}^{m} k_i,$$

$$d_{NI} = \sqrt[t]{\frac{ak_m}{bK + ak_m}},$$

$$\varepsilon = \min\left\{\sqrt[t]{\frac{ak_m}{bk_1}}, \gamma\right\}.$$

Theorem

The set $E\big(w_n(x)\big)$ is not a finite union of closed bounded intervals for $0 < x < \min\{\varepsilon, d_{NI}\}$.

Theorem

Choose $\lambda, \mu \in R^+$ *and* $s \in \mathbb{N}$ *such that every number* $\mu, \mu + \lambda, \mu + 2\lambda, \ldots, \mu + s\lambda$, *is a subsum of the (finite) series* $\sum_{i=1}^{m} k_i$, *and write*

$$d_{CI} = \sqrt[t]{\frac{b}{s \cdot a + b}}.$$

Then, whenever $d_{CI} < \varepsilon$, $E\big(w_n(x)\big)$ *contains a compact interval for any* $x \in [d_{CI}, \varepsilon)$.

Corollary *

Whenever $d_{CI} < d_{NI}$, the set $E\big(w_n(x)\big)$ is a Cantorval for any $x \in [d_{CI}, d_{NI})$.

Example. The set of subsums for the following series

$$8\sin\left(\tfrac{1}{15}\right) + 7\sin\left(\tfrac{1}{15}\right) + 6\sin\left(\tfrac{1}{15}\right) + 5\sin\left(\tfrac{1}{15}\right) + 4\sin\left(\tfrac{1}{15}\right) +$$

$$+8\sin\left(\tfrac{1}{15}\right)^2 + 7\sin\left(\tfrac{1}{15}\right)^2 + 6\sin\left(\tfrac{1}{15}\right)^2 + 5\sin\left(\tfrac{1}{15}\right)^2 + 4\sin\left(\tfrac{1}{15}\right)^2 +$$

$$\ldots$$

$$+8\sin\left(\tfrac{1}{15}\right)^n + 7\sin\left(\tfrac{1}{15}\right)^n + 6\sin\left(\tfrac{1}{15}\right)^n + 5\sin\left(\tfrac{1}{15}\right)^n + 4\sin\left(\tfrac{1}{15}\right)^n +$$

$$\ldots$$

is a Cantorval.

In such a case we have

$$k_1 = 8, k_2 = 7, k_3 = 6, k_4 = 5, k_5 = 4 \Rightarrow K = 30$$

$$\mu = 4, \lambda = 1, s = 22.$$

The function $f(x) = \sin x$ satisfies Jordan's inequality

$$\frac{2x}{\pi} < \sin x < x \quad \text{for all} \quad 0 < x < \frac{\pi}{2},$$

hence $a = 2/\pi, b = 1, t = 1$. According to the Corollary * for any x satisfying inequality

$$\frac{\pi}{44 + \pi} \leq x \leq \frac{8}{30\pi + 8}$$

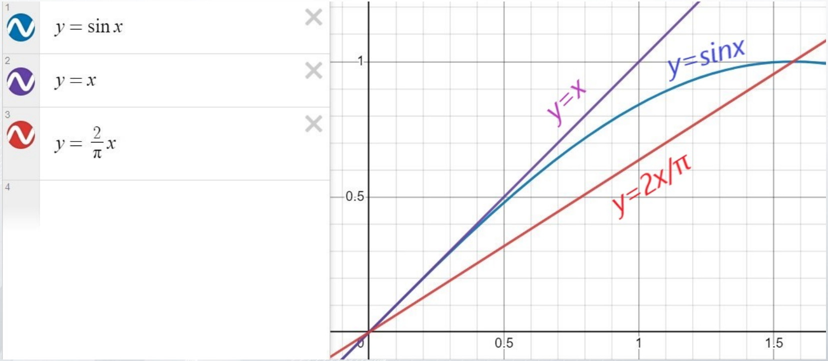
$E(w_n(x))$ is a Cantorval.

Generalised multigeometric series VIII

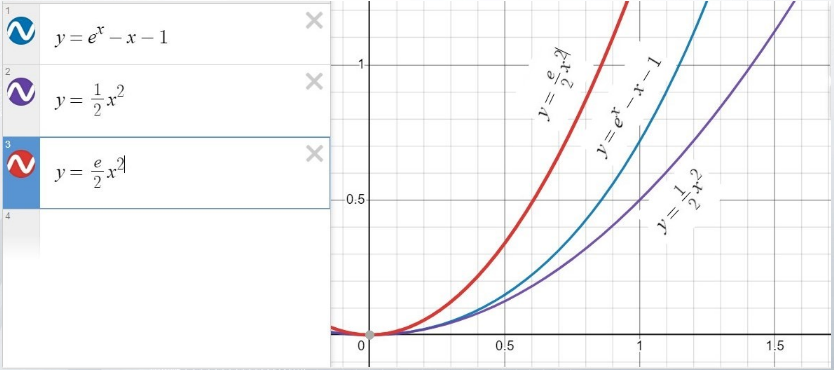
We can apply the above result for the following functions:

- $f(x) = x$ – multigeometric case;
- $f(x)$ – polynomials such that $f(0) = 0$;
- $f(x) = \sin x$;
- $f(x) = \tan x$;
- $f(x) = \ln(1 - x)$;
- $f(x) = e^x - x - 1$.

Generalised multigeometric series IX



Generalised multigeometric series X



The set of subsums of intermediate series I

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent positive series with the same type of the set of subsums (a finite union of intervals, a Cantor-type set or a Cantorval). We call the series $\sum_{n=1}^{\infty} c_n$ **intermediate** if

$$a_n < c_n < b_n, \quad \text{for all} \quad n \in \mathbb{N}.$$

Question: Under what conditions do the sets of subsums $E(a_n)$, $E(b_n)$, $E(c_n)$ share the same topological type?

Moroz M. (2024): A counterexample to the Karvatskyi–Pratsiovytyi conjecture concerning the achievement set of an intermediate series. (available at arXiv:2412.00042v1).

Open problems I

- Identifying new conditions under which a set is a Cantorval.
- Classifying homogeneous self-similar sets from a topological perspective.
- Computing the Lebesgue measure of the interior.
- Analyzing the fractal properties of the boundary.
- Studying Cantorvals in higher-dimensional spaces.

Articles devoted to the above results I

Karvatskyi, D., A. Murillo and A. Viruel (2024): The achievement set of generalized multigeometric sequences. - Results in Mathematics. 79, article no. 132.

Pratsiovytyi, M., and D. Karvatskyi (2023): Cantorvals as sets of subsums for a series connected with trigonometric functions. - Proceedings of the International Geometry Center. 15:3-4, 262–271.

Karvatskyi D., M. Pratsiovytyi and O. Makarchuk (2025): Fractal analysis of Guthrie-Nymann's set and its generalisations. to appear in the Ukr. Math. Journal. (available at arXiv:2405.16576.)

Thank you for your attention!