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Геометричні та алгебраїчні властивості бездисперсійного рівняння Нижника

111 Математика
11 Математика та статистика

Дисертація
на здобуття ступеня доктора філософії

Дисертація містить результати власних досліджень. Використання ідей, результатів і текстів інших авторів мають посилання на відповідне джерело.

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Анотація

Вінніченко О.О. Геометричні та алгебраїчні властивості бездисперсійного рівняння Нижника. — Кваліфікаційна наукова праця на правах рукопису.

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У дисертації виконано розширений симетрійний аналіз (дійсного симетричного потенціального) бездисперсійного рівняння Нижника

$$u_{txy} = (u_{xx}u_{xy})_x + (u_{xy}u_{yy})_y, \tag{1}$$

яке також називають бездисперсійним рівнянням Нижника–Новікова–Вєсєлова або навіть бездисперсійним рівнянням Новікова–Вєсєлова. Це рівняння є бездисперсійним аналогом дійсного симетричного потенціального рівняння Нижника. У наведеній повній назві рівняння атрибут "дійсне" означає, що і незалежні, і залежні змінні у рівнянні є дійсними. Вибір базового поля для змінних є важливим, оскільки від нього залежать групи точкових та контактних симетрій рівняння.

У літературі є багато спроб дослідження рівняння (1) у рамках симетрійного аналізу диференціальних рівнянь, але ці спроби зазвичай невдалі, оскільки отримані результати не є повними або достовірними. Тому було важливо виконати симетрійний аналіз бездисперсійного рівняння Нижника правильно, вичерпно й оптимально, застосовуючи широкий набір сучасних методів симетрійного аналізу та використовуючи коректну термінологію.

Разом із рівнянням (1) також розглянуто його нелінійне представлення Лакса

$$v_t = \frac{1}{3}\left(v_x^3 - \frac{u_{xy}^3}{v_x^3}\right) + u_{xx}v_x - \frac{u_{xy}u_{yy}}{v_x}, \quad v_y = -\frac{u_{xy}}{v_x} \tag{2}$$

і бездисперсійний відповідник

$$p_t = (h^1 p)_x + (h^2 p)_y, \quad h^1_y = p_x, \quad h^2_x = p_y \tag{3}$$

симетричної системи Нижника, який є потенціальною системою для рівняння (1).

У розділі 1 досліджено симетрійні властивості рівняння (1) та систем (2) і (3). Зокрема, знайдено їх максимальні алгебри ліївської інваріантності $\mathfrak{g}$, $\mathfrak{g}_{\rm L}$, $\mathfrak{g}_{\rm dN}$, а також максимальну алгебру $\mathfrak{g}_{\rm c}$ контактних симетрій рівняння (1). Вивчено структуру цих алгебр, що включає побудову достатніх для подальшого розгляду наборів їх мегаідеалів, базовими з яких є їх радикали $\mathfrak{r}$, $\mathfrak{r}_{\rm L}$, $\mathfrak{r}_{\rm dN}$. Один із необхідних мегаідеалів алгебри $\mathfrak{g}_{\rm L}$ неможливо знайти стандартними методами. Тому у дисертації розроблено новий метод пошуку мегаідеалів, який і використано у цьому випадку. Показано, що алгебра $\mathfrak{g}_{\rm c}$ є першим продовженням алгебри $\mathfrak{g}$, а алгебри $\mathfrak{g}_{\rm L}$ і $\mathfrak{g}_{\rm dN}$ — продовженнями цієї ж алгебри відповідно на псевдопотенціал v та набір потенціалів (p, q).

Застосовуючи оригінальну версію алгебраїчного методу на основі мегаідеалів, обчислено псевдогрупи точкових симетрій G, $G_{\rm L}$, $G_{\rm dN}$ відповідно для рівняння (1) та систем (2), (3), а також псевдогрупу контактних симетрій $G_{\rm c}$ рівняння (1). Виявилося, що необхідна алгебраїчна умова, яка є основою методу, повністю визначає псевдогрупу G, а тому для завершення її обчислення не потрібно використовувати прямий метод. Це перший приклад такого роду в літературі. Окрім того доведено, що псевдогрупа G містить рівно три незалежні дискретні елементи, а псевдогрупа $G_{\rm c}$ є першим продовженням псевдогрупи G. Обчислення псевдогрупи $G_{\rm c}$ є першим прикладом застосування версії алгебраїчного методу на основі мегаідеалів для знаходження псевдогрупи контактних симетрій диференціального рівняння. На відміну від неперервних точкових симетрій не всі дискретні точкові симетрії рівняння (1) можна продовжити на систему (2). Алгебраїчні частини обчислень псевдогруп $G_{\rm L}$ і $G_{\rm dN}$ схожі на їх відповідник для псевдогрупи G, але, оскільки ряд обмежень для компонент точкових перетворень симетрії не можна отримати в рамках алгебраїчного методу, то тут роль прямого методу стає суттєвішою (особливо для псевдогрупи G_{dN}), ніж у процесі побудови псевдогрупи G.

У зв'язку із зазначеною особливістю застосування алгебраїчного методу до рівняння (1) і для глибшого розуміння підґрунтя цього методу, перевірено, чи скінченновимірні підалгебри $\mathfrak{s}_1$ і $\mathfrak{s}_2$ алгебри $\mathfrak{g}$, які природним чином виникають у процесі обчислення псевдогрупи G, визначають дифеоморфізми, що стабілізують цю алгебру чи її перше продовження. Це дослідження дало несподівані результати. Зокрема, підалгебра $\mathfrak{s}_2$ визначає дифеоморфізми, що стабілізують алгебру $\mathfrak{g}$, тоді як підалгебра $\mathfrak{s}_1$ і навіть підалгебра $\bar{\mathfrak{s}}_1$, яка є природним розширенням підалгебри $\mathfrak{s}_1$ одним векторним полем з $\mathfrak{g}$, не мають цієї властивості. А перше продовження розширення підалгебри $\mathfrak{s}_2$ трьома лінійно незалежними векторними полями з $\mathfrak{g}$, що є підалгеброю алгебри $\mathfrak{g}_{\mathrm{c}} = \mathfrak{g}_{(1)}$, визначає дифеоморфізми відповідного простору струменів першого порядку, які стабілізують алгебру $\mathfrak{g}_{\mathrm{c}}$. Крім того, це дослідження містить альтернативну побудову псевдогруп G і G_{c} на основі примітивної версії алгебраїчного методу. Відповідні обчислення є набагато складнішими, ніж при використанні версії алгебраїчного методу на основі мегаідеалів, що загалом обґрунтовує її використання.

Описано всі диференціальні рівняння третього порядку з трьома незалежними змінними, які інваріантні відносно алгебри $\mathfrak{g}$. Знайдено повний набір геометричних властивостей рівняння (1), що виокремлюють його з усього класу диференціальних рівнянь із частинними похідними третього порядку з трьома незалежними змінними. Окрім інваріантності відносно алгебри $\mathfrak{g}$ він включає наявність характеристик законів збереження 1, u_{xx} і u_{yy}. Це поєднує обернену задачу групової класифікації та обернену задачу про закони збереження.

У розділі 2 вичерпно вивчено ліївські редукції рівняння (1) і побудовано широкі сім'ї його інваріантних розв'язків.

Вперше представлено точний формалізований опис повної оптимізованої процедури ліївської редукції у випадку системи рівнянь із частинними похідними з трьома незалежними змінними, релевантному для рівняння (1).

Використовуючи результати розділу 1, прокласифіковано одно- та двовимірні підалгебри алгебри $\mathfrak{g}$ і одновимірні підалгебри алгебри $\mathfrak{g}_{\rm L}$ з точністю до G- і $G_{\rm L}$-еквівалентності, відповідно. Замість стандартного підходу, що ґрунтується на знаходженні і використанні внутрішніх автоморфізмів алгебр Лі, розглянуто дію псевдогрупи G на алгебру $\mathfrak{g}$, яку знайдено через підняття векторних полів з $\mathfrak{g}$ елементами псевдогрупи G. Цей спосіб більш зручний для обчислень у випадку нескінченновимірних алгебр Лі. Крім того, при класифікації підалгебр він дозволяє враховувати не тільки неперервні, а й дискретні перетворення точкової симетрії рівняння (1), що дає можливість скоротити відповідні оптимальні списки підалгебр.

Побудовані списки підалгебр створили основу для ефективного та вичерпного виконання ліївської редукції рівняння (1) до диференціальних рівнянь із частинними похідними з двома незалежними змінними та до звичайних диференціальних рівнянь.

Під час виконання процедури ліївської редукції для рівняння (1) вперше виявлено декілька цікавих явищ. Зокрема, редуковані рівняння успадковують не всі параметри відповідних сімей нееквівалентних підалгебр. Граничним для цього явища є випадок, коли всі нееквівалентні підалгебри з сім'ї, навіть параметризованої довільними функціями, за належного вибору анзаців відповідають тому самому редукованому рівнянню. Іншим проявом цього явища є можливість відображення класу редукованих рівнянь у свій підклас, який має меншу кількість параметрів. Деякі еквівалентні двовимірні підалгебри алгебри $\mathfrak{g}$ з ненульовим одновимірним перетином індукують нееквівалентні одновимірні підалгебри максимальної алгебри ліївської інваріантності редукованого диференціального рівняння з частинними похідними, яке отримане ліївською редукцією по перетину. Алгебра $\mathfrak g$ вкладається в алгебру $\mathfrak g_{\rm L}$ через продовження векторних полів з алгебри $\mathfrak g$ на псевдопотенціал v, а тому будь-яка ліївська редукція рівняння (1) має відповідник серед ліївських редукцій системи (2), але такий відповідник загалом не єдиний навіть з точністю до $G_{\rm L}$-еквівалентності. Також, на відміну від ліївських симетрій прості та очевидні дискретні точкові симетрії рівняння (1) — навіть за оптимального вибору анзаців — можуть індукувати складні та нетривіальні дискретні точкові симетрії відповідних редукованих рівнянь.

Вперше обчислено групи точкових симетрій редукованих рівнянь, включно з їх дискретними точковими симетріями, і в усіх випадках перевірено, чи є ці симетрії або прихованими, або індукованими. Оскільки більшість розглянутих редукованих рівнянь є досить громіздкими, різні версії алгебраїчного методу набагато ефективніші для таких обчислень, ніж прямий метод. Крім того, деякі редуковані рівняння для рівняння (1) не є максимального рангу. Отже, зазначений аналіз редукованих рівнянь є, зокрема, першим в літературі явним і систематичним дослідженням ліївських та загальних точкових симетрій диференціальних рівнянь, які не є максимального рангу. Він також глибший, ніж його аналоги у більшості робіт у галузі класичного групового аналізу: застосовано ширший набір методів і технік, розв'язано незвично велику частку редукованих рівнянь і систематичніше вивчено приховані симетрії вихідного рівняння. Для інтегрування та знаходження точних розв'язків деяких редукованих звичайних диференціальних рівнянь для рівняння (1) залучено відповідні ліївські редукції системи (2). У результаті широкі сім'ї нових інваріантних розв'язків рівняння (1) побудовано у явному вигляді в термінах елементарних функцій, функцій Ламберта та гіпергеометричних функцій, а також у параметричній або неявній формах. Додатково показано, що ліївські редукції рівняння (1) до алгебраїчних рівнянь не дають нових розв'язків цього рівняння порівняно з уже побудованими.

Оскільки будь-яка функція вигляду $u = w(t,x) + \tilde{w}(t,y)$, що відповідає адитивному розділенню змінних x та y, є розв'язком рівняння (1), таке розділення змінних тривіальне для цього рівняння. Тому для пошуку неліївських розв'язків рівняння (1), які узагальнюють деякі його інваріантні розв'язки, застосовано мультиплікативне розділення змінних x та y, анзац для якого має вигляд $u = \varphi(t,x)\psi(t,y)$ з $\varphi_x \neq 0$ і $\psi_y \neq 0$. Отримані результати показують, що ще більше розв'язків рівняння (1) в деякій замкненій формі можна побудувати, використовуючи інші методи симетрійного аналізу диференціальних рівнянь.

Ключові слова: бездисперсійне рівняння Нижника, алгебра ліївської інваріантності, псевдогрупа точкових симетрій, псевдогрупа контактних симетрій, дискретна симетрія, мегаідеал, ліївські редукції, інваріантні розв'язки, приховані симетрії, нелінійне представлення Лакса, бездисперсійна система Нижника, мультиплікативне розділення змінних.

Abstract

Vinnichenko O.O. Geometric and algebraic properties of dispersionless Nizhnik equation. — Qualifying scientific work on the rights of the manuscript.

Thesis for the degree of Doctor of Philosophy, speciality 111 Mathematics. – Institute of Mathematics of NAS of Ukraine, Kyiv, 2024.

In the thesis, we carried out extended symmetry analysis of the (real symmetric potential) dispersionless Nizhnik equation

$$u_{txy} = (u_{xx}u_{xy})_x + (u_{xy}u_{yy})_y, \tag{1}$$

which is also called as the dispersionless Nizhnik–Novikov–Veselov equation or even the dispersionless Novikov–Veselov equation. This equation is the dispersionless counterpart of the real symmetric potential Nizhnik equation. In the presented full name of the equation, the attribute “real” means that both the independent and dependent variables in the equation are real. The choice of the basic field for the variables is important since the point and contact symmetry groups of the equation depend on it.

In the literature, there are many attempts to study the equation (1) within the framework of symmetry analysis of differential equations. However, they are usually unsuccessful since the obtained results are not complete or reliable. Therefore, it had been important for one to perform the symmetry analysis of the dispersionless Nizhnik equation correctly and optimally, applying a wide set of modern methods of symmetry analysis and using suitable terminology.

Simultaneously with the equation (1), we considered its nonlinear representation Lax representation

$$v_t = \frac{1}{3}\left(v_x^3 - \frac{u_{xy}^3}{v_x^3}\right) + u_{xx}v_x - \frac{u_{xy}u_{yy}}{v_x}, \quad v_y = -\frac{u_{xy}}{v_x}, \tag{2}$$

and the dispersionless counterpart

$$p_t = (h^1p)_x + (h^2p)_y, \quad h^1_y = p_x, \quad h^2_x = p_y \tag{3}$$

of the symmetric Nizhnik system.

In Chapter 1, we studied symmetry properties of the equation (1) and the systems (2) and (3). In particular, we found their maximal Lie invariance algebras $\mathfrak{g}$, $\mathfrak{g}_{\rm L}$ and $\mathfrak{g}_{\rm dN}$ and the maximal contact-symmetry algebra $\mathfrak{g}_{\rm c}$ of the equation (1).

The structure of these algebras was studied, which includes constructing the sets of their megaideals that are sufficient for further consideration, and the basic among their megaideals are their radicals $\mathfrak{r}$, $\mathfrak{r}_{\rm L}$ and $\mathfrak{r}_{\rm dN}$. One of the required megaideals of $\mathfrak{g}_{\rm L}$ cannot be found by standard methods. Therefore, we developed a new method of constructing megaideals, which was used in this case. In addition, the algebra $\mathfrak{g}_{\rm c}$ is the first prolongation of the algebra $\mathfrak{g}$, and the algebras $\mathfrak{g}_{\rm L}$ and $\mathfrak{g}_{\rm dN}$ are prolongations of $\mathfrak{g}$ to the pseudopotential v and to the tuple of potentials (p,q), respectively.

Applying an original megaideal-based version of the algebraic method, we computed the point-symmetry pseudogroups G, $G_{\rm L}$ and $G_{\rm dN}$ of the equation (1) and the systems (2) and (3), respectively, as well as the contact-symmetry pseudogroup $G_{\rm c}$ of the equation (1). It turned out that the necessary algebraic condition, which is the base of the method, completely defines the pseudogroup G, and therefore there is no need to use the direct method for completing the computation. This is the first example of this kind in the literature. In addition, we proved that the pseudogroup G contains exactly three independent discrete elements, and the pseudogroup $G_{\rm c}$ is the first prolongation of G. The computation of the pseudogroup $G_{\rm c}$ is the first example of applying the megaideal-based version of the algebraic method to finding the contact-symmetry pseudogroup of a differential equation. Unlike continuous point symmetries, not all discrete point symmetries of the equation (1) can be extended to the system (2). The algebraic parts of the computations of the pseudogroups $G_{\rm L}$

and $G_{\rm dN}$ are quite similar to their counterpart for the pseudogroup G. At the same time, since a number of restrictions for the components of point symmetry transformations cannot be derived within the framework of the algebraic method, the role of the direct method becomes more significant here (especially for the pseudogroup $G_{\rm dN}$) than in the course of constructing the pseudogroup G.

In connection with the indicated peculiarity of applying the algebraic method to the equation (1) and for a deeper understanding of the background of this method, we checked whether the finite-dimensional subalgebras $\mathfrak s_1$ and $\mathfrak s_2$ of the algebra $\mathfrak g$, which naturally arise in the course of the above computation of G, define the diffeomorphisms stabilizing this algebra or its first prolongation. This study gave unexpected results. In particular, the subalgebra $\mathfrak s_2$ defines the diffeomorphisms that stabilize $\mathfrak g$, whereas the subalgebra $\mathfrak s_1$ and even the subalgebra $\bar{\mathfrak s}_1$, which is the natural extension of the subalgebra $\mathfrak s_1$ by a vector field from $\mathfrak g$, do not have this property. Similarly, the first prolongation of the extension of the subalgebra $\mathfrak s_2$ by three linearly independent vector fields from $\mathfrak g$, which is a subalgebra of the algebra $\mathfrak g_{\rm c}=\mathfrak g_{(1)}$, defines the diffeomorphisms of the corresponding first-order jet space that stabilize $\mathfrak g_{\rm c}$. Moreover, this study contains the alternative construction of the pseudogroups G and $G_{\rm c}$ based on the primitive version of the algebraic method. The corresponding computations are much more complicated than those in the course of using the megaideal-based version of the algebraic method, which justifies the application of the latter version in general.

We described all the third-order partial differential equations in three independent variables that are invariant with respect to the algebra $\mathfrak g$. We also find a set of geometric properties of the equation (1) that singles out it from the entire class of third-order partial differential equations with three independent variables. In addition to the invariance with respect to the algebra $\mathfrak g$, it includes the presence of the conservation-law characteristics 1,

u_{xx} and u_{yy}. This combines an inverse group classification problem with an inverse problem on conservation laws.

In Chapter 2, the Lie reductions of the equation (1) are exhaustively studied and the wide families of its invariant solutions are constructed.

We presented for the first time a precise and formalized description of the complete optimized Lie reduction procedure in the case of a system of partial differential equations with three independent variables, which is relevant to the equation (1).

Using the results of Chapter 1, we classified one- and two-dimensional subalgebras of the algebra $\mathfrak{g}$ and one-dimensional subalgebras of the algebra $\mathfrak{g}_{\mathrm{L}}$ up to the G- and G_{L}-equivalences, respectively. Instead of the standard approach, which is based on finding and using inner automorphisms of Lie algebras, we considered the action of the pseudogroup G on the algebra $\mathfrak{g}$, which was found by pushing forward vector fields from $\mathfrak{g}$ by elements of the pseudogroup G. This method is more convenient for computing in the case of infinite-dimensional Lie algebras. In addition, in the course of classifying subalgebras, it allows one to take into account not only continuous, but also discrete point symmetry transformations of the equation (1), which makes it possible to reduce the corresponding optimal lists of subalgebras.

The constructed lists of subalgebras created a basis for efficiently and exhaustively carrying out Lie reductions of the equation (1) to partial differential equations with two independent variables and to ordinary differential equations.

When performing the Lie reduction procedure for the equation (1), we observed for the first time several interesting phenomena. In particular, the reduced equations inherit not all the parameters of the corresponding families of inequivalent subalgebras. The utmost for this phenomenon is the case when all inequivalent subalgebras from a family even parameterized by arbitrary functions correspond, under an appropriate choice of ansatzes, to the same reduced equation. Another display of this phenomenon is the possibility of mapping a class of reduced equations to its subclass, which has a less number of parameters. Some equivalent two-dimensional subalgebras of the algebra $\mathfrak{g}$ with a nonzero one-dimensional intersection induce inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra of a reduced partial differential equation that is obtained by the Lie reduction with respect to the intersection. The algebra $\mathfrak{g}$ is embedded in the algebra $\mathfrak{g}_{\rm L}$ via prolonging the vector fields from $\mathfrak{g}$ to the pseudopotential v, and thus any Lie reduction of the equation (1) has a counterpart among Lie reductions of the system (2) but such a counterpart is in general not unique even up to the $G_{\rm L}$-equivalence. Moreover, in contrast to Lie symmetries, simple and obvious discrete point symmetries of the equation (1), even under the optimal choice of ansatzes, can induce complicated and nontrivial discrete point symmetries of the corresponding reduced equations.

We computed for the first time the point symmetry groups of reduced equations, including their discrete point symmetries, and it was checked in all the cases whether these symmetries are hidden or induced. Since most of the obtained reduced equations for the equation (1) are quite cumbersome, various versions of the algebraic method are much more efficient in the course of the above computation than the direct method. In addition, some of these reduced equations are not of maximal rank. Therefore, the mentioned analysis of reduced equations is, in particular, the first explicit and systematic study of Lie and general point symmetries of differential equations that are not of maximal rank. It is also deeper than its analogues in most papers in the field of classical group analysis: we applied a wider set of methods and techniques, solved an unusually large proportion of reduced equations, and more systematically studied the hidden symmetries of the original equation. For integrating and finding exact solutions of some reduced ordinary differential equations for the equation (1), we involved the corresponding Lie reductions of the system (2). As a result, we constructed wide families of new invariant solutions of the equation (1) in explicit form in terms of elementary, Lambert and hypergeometric functions as well as in parametric or implicit form. In addition, we showed that Lie reductions of the equation (1) to algebraic equations give no new solutions of this equation as compared to the already constructed ones.

Since any function of the form $u = w(t, x) + \tilde{w}(t, y)$, which corresponds to the additive separation of the variables x and y, is a solution of the equation (1), this separation of variables is trivial for (1). Therefore, to look for non-Lie solutions of the equation (1) that generalize some of its invariant solutions, we used the multiplicative separation of the variables x and y, the ansatz for which has the form $u = \varphi(t, x)\psi(t, y)$ with $\varphi_x \neq 0$ and $\psi_y \neq 0$. The obtained results show that more closed-form solutions of (1) can be constructed using other tools of symmetry analysis of differential equations.

Key words: dispersionless Nizhnik equation, Lie invariance algebra, point-symmetry pseudogroup, contact-symmetry pseudogroup, discrete symmetry, megaideal, Lie reduction, invariant solutions, hidden symmetries, nonlinear Lax representation, dispersionless Nizhnik system, multiplicative separation of variables.

List of publications of PhD candidate

1. Boyko V.M., Popovych R.O. and Vinnichenko O.O., Point- and contact-symmetry pseudogroups of dispersionless Nizhnik equation, *Commun. Nonlinear Sci. Numer. Simul.* **132** (2024), 107915, 19 pp., doi:10.1016/j.cnsns.2024.107915, arXiv:2211.09759. (Scopus – Q1, WoS – Q1, SJR – Q1).

2. Vinnichenko O.O., Boyko V.M. and Popovych R.O., Lie reductions and exact solutions of dispersionless Nizhnik equation, *Anal. Math. Phys.* **14** (2024), 82, 56 pp., doi:10.1007/s13324-024-00925-y, arXiv:2308.03744. (Scopus – Q1, WoS – Q2, SJR – Q1).

3. Вінніченко О.О., Бойко В.М., Попович Р.О., Псевдогрупи точкових і контактних симетрій бездисперсійного рівняння Нижника, Тези доповідей Міжнародного симпозіуму "Симетрія та інтегровність рівнянь математичної фізики", Київ, Інститут математики НАН України, 2022, https://www.imath.kiev.ua/~appmath/Abstracts2022/Vinnichenko.html.

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Contents

Notations

$\mathfrak{a}$	a Lie algebra
$\mathcal{C}$	a class of reduced systems
G	the point-symmetry (pseudo)group of the dispersionless Nizhnik equation or the system of differential equation under consideration
G_{c}	the contact-symmetry (pseudo)group of the dispersionless Nizhnik equation
G_{dN}	the point-symmetry pseudogroup of the dispersionless Nizhnik system
G_{id}	the identity component of G
G_{L}	the point-symmetry (pseudo)group of the nonlinear Lax representation of the dispersionless Nizhnik equation
$G_{(1)}$	the first prolongation of G
$\mathfrak{g}$	the maximal Lie invariance (pseudo)algebra of the dispersionless Nizhnik equation or the system of differential equation under consideration
$\mathfrak{g}_{\mathrm{c}}$	the contact invariance (pseudo)algebra of the dispersionless Nizhnik equation
$\mathfrak{g}_{\mathrm{dN}}$	the maximal Lie invariance (pseudo)algebra of the dispersionless Nizhnik system
$\mathfrak{g}_{\mathrm{L}}$	the maximal Lie invariance (pseudo)algebra of the nonlinear Lax representation of the dispersionless Nizhnik equation
$\mathfrak{g}_{\mathrm{e}}$	the representation of $\mathfrak{g}_{\mathrm{c}}$ in evolution form
$\mathfrak{g}_{(1)}$	the first prolongation of $\mathfrak{g}$

$\mathcal{J}$, $\mathcal{J}^{\mathrm{i}}$, $\mathcal{J}^{\mathrm{s}}$	the discrete point symmetry transformations of the dispersionless Nizhnik equation
$\bar{\mathcal{J}}$, $\bar{\mathcal{J}}^{\mathrm{i}}$, $\bar{\mathcal{J}}^{v}$	the discrete point symmetry transformations of the nonlinear Lax representation of the dispersionless Nizhnik equation
$\mathcal{L}$	a system of differential equations
$\hat{\mathcal{L}}$	a reduced system
$\mathfrak{m}_j$	the megaideal of $\mathfrak{g}$, $\quad j=1,\dots,n$, $n\in\mathbb{N}$
$\bar{\mathfrak{m}}_j$	the megaideal of $\mathfrak{g}_{\mathrm{L}}$, $\quad j=1,\dots,n$, $n\in\mathbb{N}$
$\mathrm{N}_{\mathfrak{g}}(\mathfrak{s})$	the normalizer of a subalgebra $\mathfrak{s}$ of a Lie algebra $\mathfrak{g}$ in $\mathfrak{g}$
Q	a vector field
$\mathfrak{r}$	the radical of $\mathfrak{g}$
$\mathfrak{r}_{\mathrm{L}}$	the radical of $\mathfrak{g}_{\mathrm{L}}$
$\mathfrak{s}$	a finite-dimensional subalgebra of $\mathfrak{g}$
$\bar{\mathfrak{s}}$	a finite-dimensional subalgebra of $\mathfrak{g}_{\mathrm{L}}$
$\mathfrak{s}_{(1)}$	the first prolongation of $\mathfrak{s}$
Φ	a point transformation
Φ_*	the pushforward of vector fields by Φ
Ψ	a contact transformation

Introduction

Relevance of research topic. It is difficult to overestimate the importance of symmetry in life and science. In particular, symmetries are the fundamentals of various physical disciplines, including classical and quantum mechanics, relativity and particle physics. Symmetries of systems of differential equations allow one to compute exact solutions and conservation laws of these systems, and they can provide important information about whether the system under study can be integrated.

In the nineteenth century, the Norwegian mathematician Sophus Lie began to investigate continuous groups of transformations that are possessed by systems of differential equations as their symmetry groups. Thus, symmetry analysis of differential equations was established as a field of mathematics [79–84]. Lie created much of the theory of continuous point symmetries called now Lie symmetries as well as continuous contact symmetries and used it in his studies of geometry and differential equations. The research of Lie was continued by E. Noether, E.J. Cartan, L. Eisenhart, L.V. Ovsiannikov, W. Miller Jr., P. Winternitz, W.I. Fushchych, A.M. Vinogradov, N.H. Ibragimov, P.J. Olver, G.W. Bluman, S. Kumei, P.E. Hydon, S. Anco, their collaborators, followers and pupils as well as many other scientists, see, e.g., [16, 17, 20, 21, 29–33, 40, 48, 50–52, 56, 63, 65, 91, 96, 97, 103] and references therein. It is also worth to separately note the significant contribution of the Ukrainian school of group analysis of differential equations to the development of the field. This school was founded by W.I. Fushchych and includes a number of well-known and internationally recognized researches such as A.G. Nikitin, W.M. Shtelen, R.Z. Zhdanov, I.M. Tsyfra, M.I. Serov, V.I. Lahno, R.M. Cherniha, R.O. Popovych, V.M. Boyko, I.A. Yehorchenko, O.O. Vaneeva and M.O. Nesterenko, as well as their pupils, see, for instance, [1, 3, 9–13]. A number of new concepts and methods were proposed and improved, in particular, conditional symmetry [55], Q-conditional symmetry [51, 132], reduction operators and reduction modules [35, 76, 107], normalized classes of differential equations [27, 36, 71, 72, 100, 106, 112, 115], equivalence groupoids classes of systems of differential equations [38, 78, 99, 108], the megaideal-based algebraic method [26, 27, 46, 47, 85, 100], the method of furcate splitting [2, 93, 101, 113, 114], the method of mappings between classes of differential equations [102, 123], the conditional, extended and generalized extended equivalence groups [26, 106, 112], and etc.

Lie symmetries are the simplest objects related to a system $\mathcal{L}$ of differential equations in the context of group analysis of differential equations. They constitute the identity component $G_{\rm id}$ of the point-symmetry (pseudo)group G of $\mathcal{L}$, which is called the Lie symmetry (pseudo)group of $\mathcal{L}$. The infinitesimal counterpart of $G_{\rm id}$ is the maximal Lie invariance algebra $\mathfrak{g}$ of $\mathcal{L}$ consisting of the Lie-symmetry vector fields of $\mathcal{L}$ or, in other words, the generators of (local) one-parameter subgroups of G. The method for computing the (pseudo)group $G_{\rm id}$ is quite algorithmic and was originally suggested by S. Lie. Within the Lie infinitesimal approach, finding $G_{\rm id}$ reduces to finding $\mathfrak{g}$, and the latter is based on the infinitesimal invariance criterion. The application of this criterion leads to the system of determining equations for the components of Lie-symmetry vector fields of the system $\mathcal{L}$, which is a linear overdetermined system of partial differential equations and can thus often be completely integrated. Due to its algorithmic nature and realizability, the procedure of deriving such systems and solving them can be implemented using symbolic computations, and there are a number of specialized packages for this purpose in various computer algebra systems [24, 41, 43, 58, 128]. Nevertheless, at least a part of these packages sometimes miss a part of Lie symmetries, produce incorrect Lie symmetries or are even not able to derive the corresponding system of determining equations, and the situation becomes worse in the course of studying a class of systems of differential equations instead of a single system. When the algebra $\mathfrak{g}$ is computed, the (pseudo)group $G_{\rm id}$ can be constructed by solving the Lie equations with elements of $\mathfrak{g}$ and composing the obtained one-parameter subgroups. In spite of the clarity of the approach, accurately finding $G_{\rm id}$ from $\mathfrak{g}$ is in general a nontrivial problem, see the discussion on Lie symmetries of the (1+1)-dimensional linear heat equation in [72].

In addition, it is important to study Lie reductions, which give the main way to use Lie symmetries for finding exact solutions of partial differential equations [30, 32, 33, 96, 103]. Since the Lie invariance algebras of models considered in mathematical physics are usually wide enough, it is also the most universal way for constructing exact solutions of such models in general, especially, of nonlinear ones. Many papers devoted to this subject were published for several last decades but correct and comprehensive studies of Lie reductions and the corresponding reduced systems for specific systems of partial differential equations are rather exceptional, especially in the case of more than two independent variables, see, e.g., [21, 42, 44, 53, 54, 70, 71, 85, 88, 96, 105, 123], the result collections [16, 17, 20] and references therein for particular examples. The last claim is also relevant for the (real symmetric potential) dispersionless Nizhnik equation. Its classical symmetry analysis was initiated in [92], but the obtained results are neither correct nor exhaustive. This is why this analysis was still the important and interesting mathematical problem to be solved, and the present thesis is devoted to its solution.

Relation with academic programs, plans, themes, grants. The thesis was carried out at the Department of Mathematical Physics of Institute of Mathematics of National Academy of Sciences of Ukraine as a part of the research project “Symmetry and Integrability of Equations of Modern Mathematical Physics” (2020–2024, state registration number 0120U100173).

Purpose and objectives of research. *The purpose of the thesis* is to perform the extended classical symmetric analysis of the dispersionless Nizhnik equation and to study its geometric and algebraic properties.

The research object is the dispersionless Nizhnik equation jointly with its nonlinear Lax representation and the dispersionless counterpart of the symmetric Nizhnik system.

The research subject is given by the point- and contact-symmetry pseudogroups of the dispersionless Nizhnik equation, the point-symmetry pseudogroups of the corresponding nonlinear Lax representation and of the dispersionless counterpart of the symmetric Nizhnik system, the classification of one- and two-dimensional subalgebras of the maximal Lie invariance algebra of the dispersionless Nizhnik equation, the Lie reductions and the exact solutions of the real dispersionless Nizhnik equation.

Research methods. In addition to well-known methods of the theory of Lie algebras and differential equations, we used the Lie infinitesimal approach, both versions (the automorphism- and the megaideal-based ones) of the algebraic method of constructing the point-symmetry (pseudo)group of a system of differential equations, the characteristic method for constructing conservation laws of systems of differential equations, an optimized version of the Lie reduction method and the multiplicative separation of variables.

Scientific novelty of the obtained results. The main results that determine the scientific novelty of the thesis and are submitted for its defense are the following:

1. Applying an original megaideal-based version of the algebraic method, we computed the point-symmetry pseudogroups of the dispersionless Nizhnik equation, the corresponding nonlinear Lax representation and the dispersionless counterpart of the symmetric Nizhnik system.

2. Using the same method, we also constructed the contact-symmetry pseudogroup of the dispersionless Nizhnik equation, and this is the

first usage of the megaideal-based version of the algebraic method for such a construction for a differential equation.

3. It was shown that the necessary algebraic condition completely defines the point-symmetry pseudogroup of the dispersionless Nizhnik equation. This gave the first example of a system of differential equations with this property in the literature.

4. We checked whether the subalgebras of the maximal Lie invariance algebra of the dispersionless Nizhnik equation that naturally arise in the course of the above computations define the diffeomorphisms stabilizing this algebra or its first prolongation.

5. We constructed all the third-order partial differential equations in three independent variables that admit the same Lie invariance algebra as that the dispersionless Nizhnik equation. We found a set of geometric properties of this equation that exhaustively defines it.

6. The one- and two-dimensional subalgebras of the maximal Lie invariance algebra of the dispersionless Nizhnik equation are exhaustively classified, which led to the complete classification of Lie reductions of this equation.

7. Lie and point symmetries of the derived reduced equations are comprehensively studied, including the analysis of which of them correspond to hidden symmetries of the original equation. The point symmetry groups of reduced equations, in particular those that are not of maximal rank, were computed for the first time, including their discrete point symmetries.

8. The wide families of new exact invariant solutions of the dispersionless Nizhnik equation are constructed in closed form in terms of elementary, Lambert and hypergeometric functions as well as in parametric or implicit form.

9. Multiplicative separation of variables was used for illustrative construction of non-invariant solutions of the dispersionless Nizhnik equation.

Practical significance of the obtained results. The thesis is theoretical in its essence. The obtained results are new. They can be used in further studies of various differential equations of mathematical physics that arise in real-world applications.

Personal contribution of the PhD candidate. The results presented in the thesis were obtained by the PhD candidate independently. In the co-authored papers [39, 127], R.O. Popovych was responsible for determining the research direction and posing the problems to be studied, verifying the obtained results and the proofreading of the papers was entrusted to V.M. Boyko.

Approbation of the thesis results. The main results of the thesis were reported and discussed at:

- Seminar of Department of Mathematical Physics of Institute of Mathematics of National Academy of Sciences of Ukraine (Kyiv, 2022–2024);
- International Symposium "Symmetry and Integrability of Equations of Mathematical Physics" (Kyiv, Institute of Mathematics of NAS of Ukraine, 2022);
- International Conference of Young Mathematicians (Kyiv, Institute of Mathematics of NAS of Ukraine, 2023);
- Workshop CDSS (Complex Dynamical Systems in the Science): theory, mathematical modelling, computing and application (Kyiv, Institute of Mathematics of NAS of Ukraine, 2023);
- Seminar of Young Scientists (Kyiv, Institute of Mathematics of NAS of Ukraine, 2024);
- XII All-Ukrainian Scientific Conference of Young Mathematicians (Kyiv, National University of Kyiv Mohyla Academy, 2024);

- Conference of Young Mathematicians "Pidstryhach readings – 2024" (Lviv, Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of NAS of Ukraine, 2024);
- International Scientific Online Conference "Algebraic and Geometric Methods of Analysis" (Odesa, Odesa National University of Technology, 2024);
- Bogolyubov Kyiv Conference "Problems of Theoretical and Mathematical Physics" (Kyiv, Institute of Mathematics of NAS of Ukraine, 2024).

Publications. The results of the thesis were published in nine scientific publications, two of them [39, 127] are in journals from Q1 (according to the classification of SCImago Journal & Country Rank) that together are equated to four publications. Seven publications [4–8, 125, 126] are abstracts of PhD candidate's talks at international and all-Ukrainian scientific conferences and workshops.

Structure and volume of thesis. The thesis contains annotations in Ukrainian and English, a list of the author's publications, acknowledgments, contents, notations, an introduction, two chapters, a conclusion, a list of references that contains 132 items and one appendix. The total volume of the thesis is 171 pages, of which the list of references and the appendix take 15 and 4 pages, respectively.

Chapter 1

Point- and contact-symmetry pseudogroups of dispersionless Nizhnik equation

The entire point-symmetry (resp. contact-symmetry) (pseudo)group G of the system $\mathcal{L}$ cannot be constructed within the framework of the infinitesimal approach. Since finding $\mathfrak{g}$ and then $G_{\rm id}$ from $\mathfrak{g}$ is a much simpler problem than finding the entire G, the latter problem can be assumed to be equivalent to the construction of a complete set of discrete point symmetry transformations of the system $\mathcal{L}$ that are independent up to composing with each other and with continuous point symmetry transformations of $\mathcal{L}$.[1.1] The only universal tool for the above constructions is the direct method based on the definition of point symmetry transformation and the chain rule [27, 71, 72, 100]. The technique of its usage is similar to that of the infinitesimal method, see [66] for technical details of more general computations of admissible (or form-preserving) transformations in classes of systems of differential equations in the case of two independent variables and one dependent variable. At the same time, the application of the direct method to the system $\mathcal{L}$ leads to a nonlinear overdetermined system of partial differential equations for the components of point symmetry transformations, which is much more difficult to solve than its counterpart for Lie symmetries. This is why a number of special techniques

[1.1] Often, such a complete set can be chosen to consist of simple discrete point symmetry transformations, which can be guessed straightforwardly from the form of $\mathcal{L}$. A quite common technique in the literature is to consider a (pseudo)subgroup of G jointly generated by the elements of $G_{\rm id}$ and the guessed discrete point symmetry transformations, and such a subgroup may coincide with the entire G. The problem is to prove that this is the case or to find missed independent discrete point symmetry transformations.

within the framework of the direct method were developed for simplifying related computations, including switching between the original and the transformed variables, mapping the system $\mathcal{L}$ under study to a more convenient one and preliminarily finding the equivalence (pseudo)group of a normalized class of systems of differential equations that contains the system $\mathcal{L}$ [27, 36, 71, 72].

A more sophisticated and systematic method for this purpose was first suggested by Hydon [60–63]. It works in the case when the maximal Lie invariance algebra $\mathfrak{g}$ of the system $\mathcal{L}$ is nonzero and finite-dimensional, and it is based on the fact that the pushforward Φ_* of $\mathfrak{g}$ by any element Φ of the group G is an automorphism of $\mathfrak{g}$. Chosen a basis $(Q^1,\dots,Q^n)$ of $\mathfrak{g}$, where $n=\dim\mathfrak{g}$, this condition is equivalent to

$$\Phi_*Q^i=\sum_{j=1}^n a_{ji}Q^j,\quad i=1,\dots,n,$$

where $(a_{ji})_{i,j=1,\dots,n}$ is the matrix of an automorphism of $\mathfrak{g}$ in this basis. Finding the general form of automorphism matrices and splitting the last condition componentwise, one derives a system $\mathrm{DE_a}(\mathcal{L})$ of determining equations for the components of an arbitrary point symmetry transformation Φ of $\mathcal{L}$. The system $\mathrm{DE_a}(\mathcal{L})$ is a *linear* and, if $n>1$, overdetermined system of partial differential equations but, in general, it does not define the group G completely. After integrating this system, one should continue the computation within the framework of the direct method using the derived expressions for components of Φ, which essentially simplifies the application of the direct method in total. Due to involving algebraic conditions, we call the above procedure the *algebraic method of constructing the point-symmetry (pseudo)group of a system of differential equations.* The algebraic approach was extended in [27] to the case when the maximal Lie invariance algebra $\mathfrak{g}$ is infinite-dimensional via replacing Hydon's condition with the weaker condition that $\Phi_*\mathfrak{m}\subseteq\mathfrak{m}$ for any megaideal $\mathfrak{m}$

of $\mathfrak{g}$.[1.2] To distinguish Hydon's and our versions of the algebraic method from each other, we shortly call them the *automorphism-based* and the *megaideal-based* methods, respectively. In principle, one can use the *primitive version of the algebraic method* that is based only on the condition $\Phi_*\mathfrak{g} \subseteq \mathfrak{g}$ and involves no knowledge of automorphisms or megaideals of $\mathfrak{g}$. Nevertheless, the primitive version of the algebraic method leads to much more cumbersome computations than its more sophisticated counterparts, see discussions below.

Analogs of both these methods for finding equivalence (pseudo)groups of classes of differential equations or, equivalently, their discrete equivalence transformations were suggested in [26]. The automorphism-based method was strengthened in [70] for the case of nonsolvable finite-dimensional maximal Lie invariance algebras via effectively involving the Levi–Malcev theorem and results on automorphisms of semisimple Lie algebras. The megaideal-based method was developed and applied to several important systems of differential equations [46, 47, 85, 100]. An essential part of this development was the invention of new techniques for constructing megaideals of a Lie algebra without knowing its automorphism group, which was initiated in [111] and continued in [26, 27, 46]. The megaideal- and automorphism-based methods were combined in [46]. In the course of computing the point-symmetry group of the Boiti–Leon–Pempinelli system in [85], a special version of the megaideal-based method was suggested, whose basic condition is $\Phi_*(\mathfrak{m} \cap \mathfrak{s}) \subseteq \mathfrak{m}$ for a selected finite-dimensional subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ and any megaideal $\mathfrak{m}$ of $\mathfrak{g}$ from a constructed collection of such megaideals, and this is the method that is applied below.

In the case of a system $\mathcal{L}$ with one dependent variable, contact symmetries of $\mathcal{L}$ can be studied analogously, see [61] for the corresponding

[1.2] Recall that a *megaideal* $\mathfrak{m}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ that is invariant with respect to any transformation $\mathfrak{T}$ from the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$, $\mathfrak{T}\mathfrak{m} \subseteq \mathfrak{m}$ [26, 111]. Another name for $\mathfrak{m}$ is a *fully characteristic ideal* of $\mathfrak{g}$ [59, Exercise 14.1.1]. Since $\mathfrak{T}^{-1} \in \mathrm{Aut}(\mathfrak{g})$ for any $\mathfrak{T} \in \mathrm{Aut}(\mathfrak{g})$, simultaneously with the invariance condition $\mathfrak{T}\mathfrak{m} \subseteq \mathfrak{m}$ we also have $\mathfrak{T}^{-1}\mathfrak{m} \subseteq \mathfrak{m}$ and hence in fact $\mathfrak{T}\mathfrak{m} = \mathfrak{m}$. Each megaideal of $\mathfrak{g}$ is an ideal and, moreover, a characteristic ideal of $\mathfrak{g}$.

automorphism-based method. More specifically, let $\mathfrak{g}_{\rm c}$ and $G_{\rm c}$ denote the contact Lie invariance algebra of the system $\mathcal{L}$ and its contact-symmetry (pseudo)group, respectively. One should first compute the algebra $\mathfrak{g}_{\rm c}$ within the framework of the infinitesimal approach and then use the condition that the pushforward of $\mathfrak{g}_{\rm c}$ by any element Ψ of $G_{\rm c}$ is an automorphism of $\mathfrak{g}_{\rm c}$. In the course of this computation, the contact condition should be taken into account as well, see item (ii) of the proof of Theorem 1.3 below. In a similar way, one can also compute the contact equivalence (pseudo)group of a class of systems of differential equations with one dependent variable.

The initial inspiration of the paper [39], which is the source of this chapter, was to enhance results of [92] and, applying the original megaideal-based version of the algebraic method from [85], to present a correct and complete computation of the point- and contact-symmetry pseudogroups G and $G_{\rm c}$ of the dispersionless counterpart

$$u_{txy} = (u_{xx}u_{xy})_x + (u_{xy}u_{yy})_y \tag{1.1}$$

of the (real symmetric potential) Nizhnik equation for the (real) Nizhnik system [94, Eq. (4)], which we call the dispersionless Nizhnik equation. It explicitly appeared for the first time in an equivalent form in [67, Eq. (63)], where it was called the dispersionless Nizhnik–Novikov–Veselov equation due to [94, Eq. (4)] and the later paper [124, Eq. (5)]. It is also known as the dispersionless Novikov–Veselov equation (see, e.g., [104, Eq. (5)] and [92, Eq. (1)]). The proper Novikov–Veselov counterpart of (1.1) was derived in [68, Eq. (30)] and [69, Eq. (32)] as a model of nonlinear geometrical optics. More specifically, it is the equation for the refractive index under the geometrical optics limit of the Maxwell equations for certain nonlinear media with slow variation along one axis and particular dependence of the dielectric constant on frequency and fields.

Remark 1.1. The symmetric and asymmetric (potential) Nizhnik equations are obtained via introducing potentials in the symmetric and asymmetric cases of the system (4) from [94],

$$w_t = k_1 w_{xxx} + k_2 w_{yyy} + 3(v^1 w)_x + 3(v^2 w)_y, \quad v^1_y = k_1 w_x, \quad v^2_x = k_2 w_y,$$

where both parameters k_1 and k_2 are nonzero or one (and only one) of them is equal to zero and thus they are reduced by scale equivalence transformations to $(k_1, k_2) = (1, 1)$ or $(k_1, k_2) = (1, 0)$, respectively. The asymmetric Nizhnik equation is also called the Boiti–Leon–Manna–Pempinelli equation due to [34]. Both the Nizhnik equations can be considered under the assumptions that all the independent and dependent variables are either real (the real Nizhnik equation) or complex (the complex Nizhnik equation) or the unknown function is a complex-valued function of real independent variables (the partially complexified Nizhnik equation). A specific version of the symmetric Nizhnik equation, where the independent variables are the complex conjugates of each other and the principal unknown function is real, was given by Novikov and Veselov in [124, Eq. (5)]. The dispersionless counterpart of the Novikov–Veselov system takes the form

$$v_t = (wv)_z + (\bar{w}v)_{\bar{z}}, \quad w_{\bar{z}} = -3v_z,$$

where $z = x+\mathrm{i}y$, $\bar{z} = x-\mathrm{i}y$, $\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$, $w = w^1 + \mathrm{i}w^2$, and v, w^1 and w^2 are real-valued functions of the real variables (t, x, y). Introducing potentials reduces it to the equation

$$\triangle u_t = \frac{1}{2}((u_{yy} - u_{xx})\triangle u)_x + (u_{xy}\triangle u)_y,$$

where $\triangle u := u_{xx} + u_{yy}$ and u is a real-valued function of (t, x, y). The point which fields (real or complex) are run by the independent and dependent variables is often not specified in the literature but, in fact, it is essential in the course of computing point and contact symmetries. In this chapter, we study the real dispersionless Nizhnik equation, which is the dispersionless counterpart of the real symmetric potential Nizhnik equation.

Although the correct descriptions of the pseudogroups G and G_c are of interest by themselves, the main value of these results is another. They give the first examples of using the algebraic method in the literature, where the Hydon's condition or its weakened version involving megaideals exhaustively define the corresponding point- and contact-symmetry (pseudo)groups, making the direct parts of computing trivial. Moreover, in the course of showing that the pseudogroup $G_{\rm c}$ coincides with the first prolongation of the pseudogroup G, we first apply the megaideal-based version of the algebraic method to finding the contact-symmetry (pseudo)group of a partial differential equation. To optimize the computation of the point-symmetry pseudogroup $G_{\rm L}$ of the nonlinear Lax representation (1.14) of the equation (1.1), we invent a new technique for computing megaideals of Lie algebras, which allows us to construct one more megaideal of the maximal Lie invariance algebra $\mathfrak g_{\rm L}$ of (1.14) in addition to those that can be found with known techniques.

For a deeper understanding of the background of the algebraic method, we check whether the subalgebras of the maximal Lie invariance algebra $\mathfrak g$ of the equation (1.1) that naturally arise in the course of the above computation of G define the diffeomorphisms stabilizing this algebra. The same property is also studied for several subalgebras of the contact invariance algebra $\mathfrak g_{\rm c}$ of (1.1), which coincides with the first prolongation $\mathfrak g_{(1)}$ of the algebra $\mathfrak g$. This study gives unexpected results and, moreover, contains alternative constructions of the pseudogroups G and $G_{\rm c}$ based on the primitive version of the algebraic method. The corresponding computations are much more complicated than those in the course of using the megaideal-based method, which nicely justifies the application of the latter method in general.

Since the maximal Lie invariance algebra $\mathfrak g$ of the equation (1.1) completely defines its point-symmetry group G by means of the condition $\Phi_*\mathfrak g\subseteq\mathfrak g$ for any $\Phi\in G$, the natural question is whether this algebra defines the equation (1.1) itself as well. In other words, given a single third-order partial differential equation possessing $\mathfrak g$ as its Lie invariance algebra, does this equation necessarily coincide with the equation (1.1)? We show that this is not the case but the answer becomes positive if the $\mathfrak{g}$-invariance is supplemented with the condition of admitting the conservation-law characteristics 1, u_{xx} and u_{yy}. This combines an inverse group classification problem (see, e.g., [103, p. X], [97, pp. 191–199] and [109, Section II.A]) with an inverse problem on conservation laws [110]. Therefore, we find a nice set of geometric properties of the equation (1.1) that exhaustively defines it, see [22,57,73,86,87,95,120,121] and references therein on similar studies. Since $\mathfrak{g}_{\rm c}=\mathfrak{g}_{(1)}$, we can reformulate the corresponding assertion, replacing Lie symmetries with contact ones. As a by-product, we describe all the third-order partial differential equations in three independent variables that are invariant with respect to the algebra $\mathfrak{g}$.

The results of Chapter 1 were presented in the paper [39] and in the abstracts of conference talks [4, 5, 8, 125, 126].

1.1. Structure of Lie invariance algebra

The maximal Lie invariance (pseudo)algebra $\mathfrak{g}$ of the dispersionless Nizhnik equation (1.1) is infinite-dimensional and is spanned by the vector fields

$$
\begin{gathered}
D^t(\tau)=\tau\partial_t+\tfrac13\tau_tx\partial_x+\tfrac13\tau_ty\partial_y-\tfrac1{18}\tau_{tt}(x^3+y^3)\partial_u,\\
D^{\rm s}=x\partial_x+y\partial_y+3u\partial_u,\\
P^x(\chi)=\chi\partial_x-\tfrac12\chi_tx^2\partial_u,\quad P^y(\rho)=\rho\partial_y-\tfrac12\rho_ty^2\partial_u,\\
R^x(\alpha)=\alpha x\partial_u,\quad R^y(\beta)=\beta y\partial_u,\quad Z(\sigma)=\sigma\partial_u,
\end{gathered}
\tag{1.2}
$$

where τ, χ, ρ, α, β and σ run through the set of smooth functions of t, cf. [92]. Moreover, the contact invariance (pseudo)algebra $\mathfrak{g}_{\rm c}$ of the equation (1.1) coincides with the first prolongation $\mathfrak{g}_{(1)}$ of the algebra $\mathfrak{g}$, and generalized symmetries of this equation at least up to order five are exhausted, modulo the equivalence of generalized symmetries, by its Lie symmetries. We recomputed the algebra $\mathfrak{g}$ as well as first computed the algebras $\mathfrak{g}_{\rm L}$

and $\mathfrak{g}_{\rm dN}$ (see Sections 1.5 and 1.6) using the command Infinitesimals of the built-in Maple package PDEtools and the packages DESOLV [41, 128] and Jets [24, 89] for Maple; the latter package was also used for computing the algebra $\mathfrak{g}_{\rm c}$ and generalized symmetries of (1.1) up to order five.

Up to the antisymmetry of the Lie bracket, the nonzero commutation relations between the vector fields (1.2) spanning $\mathfrak{g}$ are exhausted by

$$
\begin{aligned}
&[D^t(\tau^1), D^t(\tau^2)] = D^t(\tau^1\tau^2_t - \tau^1_t\tau^2),\\
&[D^t(\tau), P^x(\chi)] = P^x\big(\tau\chi_t - \tfrac13\tau_t\chi\big),\\
&[D^t(\tau), P^y(\rho)] = P^y\big(\tau\rho_t - \tfrac13\tau_t\rho\big),\\
&[D^t(\tau), R^x(\alpha)] = R^x\big(\tau\alpha_t + \tfrac13\tau_t\alpha\big),\\
&[D^t(\tau), R^y(\beta)] = R^y\big(\tau\beta_t + \tfrac13\tau_t\beta\big),\\
&[D^t(\tau), Z(\sigma)] = Z(\tau\sigma_t),\\
&[D^{\rm s}, P^x(\chi)] = -P^x(\chi), \quad [D^{\rm s}, P^y(\rho)] = -P^y(\rho),\\
&[D^{\rm s}, R^x(\alpha)] = -2R^x(\alpha), \quad [D^{\rm s}, R^y(\beta)] = -2R^y(\beta),\\
&[D^{\rm s}, Z(\sigma)] = -3Z(\sigma),\\
&[P^x(\chi^1), P^x(\chi^2)] = -R^x(\chi^1\chi^2_t - \chi^1_t\chi^2),\\
&[P^y(\rho^1), P^y(\rho^2)] = -R^y(\rho^1\rho^2_t - \rho^1_t\rho^2),\\
&[P^x(\chi), R^x(\alpha)] = Z(\chi\alpha), \quad [P^y(\rho), R^y(\beta)] = Z(\rho\beta).
\end{aligned}
\tag{1.3}
$$

We find megaideals of the algebra $\mathfrak{g}$ that will be used for computing the point-symmetry pseudogroup G of the equation (1.1). The only megaideal that is obvious in view of the above commutation relations is

$$\mathfrak{m}_1 := \mathfrak{g}' = \big\langle D^t(\tau), P^x(\chi), P^y(\rho), R^x(\alpha), R^y(\beta), Z(\sigma)\big\rangle.$$

Here and throughout the thesis $\mathfrak{z}(\mathfrak{s})$ and $\mathfrak{s}'$ denote the center and the derived algebra of a subalgebra $\mathfrak{s}$ of algebra $\mathfrak{g}$, respectively, $\mathfrak{s}'' = (\mathfrak{s}')'$, $\mathfrak{s}''' = (\mathfrak{s}'')'$ and $\mathfrak{s}^3 := [\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]]$. More generally, $s^{(0)} := \mathfrak{s}$ and the nth derived algebra $\mathfrak{s}^{(n)}$ of $\mathfrak{s}$ is recursively defined by $\mathfrak{s}^{(n+1)} = (\mathfrak{s}^{(n)})'$, $n \in \mathbb{N}$.

The computation of other megaideals of the algebra $\mathfrak{g}$ is based on the following assertion.

Lemma 1.2. *The radical* $\mathfrak{r}$ *of* $\mathfrak{g}$ *coincides with*

$$\big\langle D^{\rm s}, P^x(\chi), P^y(\rho), R^x(\alpha), R^y(\beta), Z(\sigma)\big\rangle.$$

Proof. Following the proof of Lemma 1 in [85], we denote the span from lemma's statement by $\mathfrak{s}$. To conclude that it coincides with the radical $\mathfrak{r}$ of $\mathfrak{g}$, we prove that it is the maximal solvable ideal of $\mathfrak{g}$.

The commutation relations between the vector fields spanning $\mathfrak{g}$, see (1.3), imply that $\mathfrak{s}$ is an ideal of $\mathfrak{g}$. Since the fourth derived algebra $\mathfrak{s}^{(4)}$ of $\mathfrak{s}$ is equal to $\{0\}$, then the ideal $\mathfrak{s}$ is solvable (of solvability rank four).

Now we show that the solvable ideal $\mathfrak{s}$ of $\mathfrak{g}$ is maximal in $\mathfrak{g}$. Let $\mathfrak{s}_1$ be an ideal of $\mathfrak{g}$ properly containing $\mathfrak{s}$. This means that at least for one nonvanishing value τ^1 of the parameter function τ, the corresponding vector field $D^t(\tau^1)$ belongs to $\mathfrak{s}_1$. Denote by I an interval in the domain of τ^1 such that $\tau^1(t)\neq 0$ for any $t\in I$. We restrict all the parameter functions in $\mathfrak{g}$ on the interval I. Since $\mathfrak{s}_1$ is an ideal of $\mathfrak{g}$, the commutator $[D^t(\tau), D^t(\tau^1)] = D^t(\tilde\tau)$ with $\tilde\tau := \tau\tau^1_t - \tau_t\tau^1$ belongs to $\mathfrak{s}_1$ for all $\tau\in {\rm C}^\infty(I)$. If the function τ runs through ${\rm C}^\infty(I)$, then, in view of the existence theorem for first-order linear ordinary differential equations, the function $\tilde\tau$ also runs through ${\rm C}^\infty(I)$. Therefore, $\mathfrak{s}_1\supset\langle D^t(\tau)\rangle$ and thus the nth derived algebra $\mathfrak{s}_1^{(n)}$ of $\mathfrak{s}_1$ contains $\langle D^t(\tau)\rangle\neq\{0\}$ for any $n\in\mathbb{N}$ as well, i.e., the ideal $\mathfrak{s}_1$ is not solvable. Hence the span $\mathfrak{s}$ is maximal as a solvable ideal of $\mathfrak{g}$. □

We set $\mathfrak{m}_2 := \mathfrak{r}$. In view of properties of megaideals [26, 111], it is easy to construct several other megaideals of the algebra $\mathfrak{g}$,

$$\begin{aligned}
&\mathfrak{m}_3 := \mathfrak{m}_2' = \mathfrak{m}_1\cap\mathfrak{m}_2 = \big\langle P^x(\chi), P^y(\rho), R^x(\alpha), R^y(\beta), Z(\sigma)\big\rangle,\\
&\mathfrak{m}_4 := \mathfrak{m}_2'' = \big\langle R^x(\alpha), R^y(\beta), Z(\sigma)\big\rangle,\\
&\mathfrak{m}_5 := (\mathfrak{m}_3)^3 = \mathfrak{z}(\mathfrak{m}_3) = \big\{Z(\sigma)\big\},\\
&\mathfrak{m}_6 := \mathfrak{z}(\mathfrak{m}_1) = \big\langle Z(1)\big\rangle.
\end{aligned}$$

Overall, the algebra $\mathfrak{g}$ contains the proper megaideal $\mathfrak{m}_1=\mathfrak{g}'$ and the chain of proper megaideals contained in its radical,

$$\mathfrak{g}\supsetneq\mathfrak{r}=:\mathfrak{m}_2\supsetneq\mathfrak{m}_3\supsetneq\mathfrak{m}_4\supsetneq\mathfrak{m}_5\supsetneq\mathfrak{m}_6.$$

Each of them is essential when applying the algebraic method to construct the point-symmetry pseudogroup of the dispersionless Nizhnik equation (1.1) in the sense that it is not the sum of other proper megaideals. Note that in contrast to the megaideals $\mathfrak{m}_j$, $j=1,\dots,6$, the improper nonzero megaideal, which is the algebra $\mathfrak{g}$ itself, is not essential in this sense since $\mathfrak{g}=\mathfrak{m}_1+\mathfrak{m}_2$. Among the constructed proper megaideals, only the megaideal $\mathfrak{m}_6$ is finite-dimensional and, moreover, it is one-dimensional. It is clear that within the above elementary consideration, we cannot answer the question of whether the megaideals $\mathfrak{m}_j$, $j=1,\dots,6$, exhaust the entire set of proper megaideals of the (infinite-dimensional) algebra $\mathfrak{g}$.

Since $\mathfrak{g}_{\rm c}=\mathfrak{g}_{(1)}\simeq\mathfrak{g}$, all the above claims on the structure of the algebra $\mathfrak{g}$ are also relevant for the algebra $\mathfrak{g}_{\rm c}$ after reformulating them for the corresponding first prolongations, which are marked by the additional subscript "(1)".

1.2. Point- and contact-symmetry pseudogroups

Theorem 1.3. *(i) The point-symmetry pseudogroup G of the dispersionless Nizhnik equation (1.1) is generated by the transformations of the form*

$$\begin{aligned}&\tilde t=T(t),\quad \tilde x=CT_t^{1/3}x+X^0(t),\quad \tilde y=CT_t^{1/3}y+Y^0(t),\\ &\tilde u=C^3u-\frac{C^3T_{tt}}{18T_t}(x^3+y^3)-\frac{C^2}{2T_t^{1/3}}(X^0_tx^2+Y^0_ty^2)\\ &\qquad+W^1(t)x+W^2(t)y+W^0(t)\end{aligned}\tag{1.4}$$

and the transformation $\mathcal{J}$: $\tilde t=t$, $\tilde x=y$, $\tilde y=x$, $\tilde u=u$. Here T, X^0, Y^0, W^0, W^1 and W^2 are arbitrary smooth functions of t with $T_t\neq0$, and C is an arbitrary nonzero constant.

(ii) The contact-symmetry pseudogroup $G_{\rm c}$ of the dispersionless Nizhnik equation (1.1) *coincides with the first prolongation $G_{(1)}$ of the pseudogroup G.*

Proof. (i) Since the maximal Lie invariance algebra $\mathfrak{g}$ of the equation (1.1) is infinite-dimensional, we compute the pseudogroup G using the modification of the megaideal-based method that was suggested in [85]. The application of this method to the equation (1.1) is based on the following observation. If a point transformation Φ in the space with the coordinates (t,x,y,u),

$$\Phi\colon\ (\tilde t,\tilde x,\tilde y,\tilde u)=(T,X,Y,U),$$

where (T,X,Y,U) is a tuple of smooth functions of (t,x,y,u) with nonvanishing Jacobian, is a point symmetry of the equation (1.1), then $\Phi_*\mathfrak{m}_j\subseteq\mathfrak{m}_j$, $j=1,\dots,6$. Here and in what follows Φ_* denotes the pushforward of vector fields by Φ, and $z\in\{x,y\}$.

We choose the following linearly independent elements of $\mathfrak{g}$:

$$\begin{aligned}
&Q^1:=Z(1),\quad Q^2:=Z(t),\quad Q^{3z}:=R^z(1),\\
&Q^{4z}:=P^z(1),\quad Q^{5z}:=P^z(t),\quad Q^6:=D^{\rm s},\\
&Q^7:=D^t(1),\quad Q^8:=D^t(t).
\end{aligned}$$

Since $Q^1\in\mathfrak{m}_6$, $Q^2\in\mathfrak{m}_5$, $Q^{3z}\in\mathfrak{m}_4$, $Q^{4z},Q^{5z}\in\mathfrak{m}_3$, $Q^6\in\mathfrak{m}_2$ and $Q^7,Q^8\in\mathfrak{m}_1$, then

$$\begin{aligned}
&\Phi_*Q^i=\tilde Z(\tilde\sigma^i),\quad i=1,2,\\
&\Phi_*Q^{iz}=\tilde R^x(\tilde\alpha^{iz})+\tilde R^y(\tilde\beta^{iz})+\tilde Z(\tilde\sigma^{iz}),\quad i=3,\\
&\Phi_*Q^{iz}=\tilde P^x(\tilde\chi^{iz})+\tilde P^y(\tilde\rho^{iz})+\tilde R^x(\tilde\alpha^{iz})+\tilde R^y(\tilde\beta^{iz})\\
&\qquad\qquad+\tilde Z(\tilde\sigma^{iz}),\quad i=4,5,\\
&\Phi_*Q^i=\lambda^i\tilde D^{\rm s}+\tilde P^x(\tilde\chi^i)+\tilde P^y(\tilde\rho^i)+\tilde R^x(\tilde\alpha^i)+\tilde R^y(\tilde\beta^i)\\
&\qquad\qquad+\tilde Z(\tilde\sigma^i),\quad i=6,\\
&\Phi_*Q^i=\tilde D^t(\tilde\tau^i)+\tilde P^x(\tilde\chi^i)+\tilde P^y(\tilde\rho^i)+\tilde R^x(\tilde\alpha^i)+\tilde R^y(\tilde\beta^i)\\
&\qquad\qquad+\tilde Z(\tilde\sigma^i),\quad i=7,8.
\end{aligned}\tag{1.5}$$

Here λ^6 and $\tilde\sigma^1$ are constants, the other parameters are smooth functions of $\tilde t$, and $\tilde\sigma^1\tilde\sigma^2\neq 0$.

We will simultaneously present two slightly different proofs, respectively using elements with $i\in\{1,\dots,6\}$ or with $i\in\{1,\dots,5,7,8\}$. For each relevant i and for each $z\in\{x,y\}$ whenever it is relevant, we expand the corresponding equation from (1.5), split it componentwise and pull the result back by Φ. We simplify the obtained constraints, taking into account constraints derived in the same way for preceding values of i and omitting the constraints satisfied identically in view of other constraints.

Thus, for $i=1,2$, we get

$$T_u=X_u=Y_u=0,\quad U_u=\tilde\sigma^1,\quad tU_u=\tilde\sigma^2(T).$$

Since $\tilde\sigma^1\neq 0$, this implies $U=U^1u+U^0(t,x,y)$ with constant $U^1\neq 0$ and $t=f(T)$ with $f(\tilde t):=\tilde\sigma^2(\tilde t)/\tilde\sigma^1$. Differentiating the equality $t=f(T)$ with respect to t gives $1=f_{\tilde t}(T)T_t$. Therefore, the derivative $f_{\tilde t}$ does not vanish, and according to the inverse function theorem, we obtain that $T=T(t)$ with $T_t\neq 0$ since the Jacobian of Φ does not vanish.

Using the same procedure for $i=3$ results in the equations

$$\begin{aligned} xU^1&=\tilde\alpha^{3x}(T)X+\tilde\beta^{3x}(T)Y+\tilde\sigma^{3x}(T),\\ yU^1&=\tilde\alpha^{3y}(T)X+\tilde\beta^{3y}(T)Y+\tilde\sigma^{3y}(T).\end{aligned}\tag{1.6}$$

The matrix constituted by the coefficients of (X,Y) in the system (1.6) is nondegenerate since otherwise this system would imply a nonidentity affine constraint for (x,y) with coefficients depending at most on t. Solving the system (1.6) with respect to (X,Y) leads to the representation

$$X=X^1(t)x+X^2(t)y+X^0(t),\quad Y=Y^1(t)x+Y^2(t)y+Y^0(t),$$

where $X^1Y^2-X^2Y^1\neq 0$ due to nonvanishing the Jacobian of Φ. We will also need the counterpart of this representation that is solved with respect to (x,y),

$$\begin{aligned} x&=\tilde X^1(t)X+\tilde X^2(t)Y+\tilde X^0(t),\\ y&=\tilde Y^1(t)X+\tilde Y^2(t)Y+\tilde Y^0(t),\end{aligned}\tag{1.7}$$

where $\tilde X^1\tilde Y^2-\tilde X^2\tilde Y^1\neq0$ as well, and

$$\begin{pmatrix}\tilde X^1 & \tilde X^2\\ \tilde Y^1 & \tilde Y^2\end{pmatrix}=\begin{pmatrix}X^1 & X^2\\ Y^1 & Y^2\end{pmatrix}^{-1}.$$

Instead of the equations from (1.5) with $i=4,5$, we first immediately consider their combinations. More specifically, for each z we subtract the equation with $i=4$ multiplied by t from the equation with $i=5$ and with the same z. In the obtained equations, we collect the $\tilde u$-components, pull them back by Φ, substitute the expressions (1.7) for (x,y) into them and then collect the coefficients of XY. The splitting with respect to (X,Y) is allowed here due to the functional independence of t, X and Y. In view of the inequality $U^1\neq0$, this leads to the equations $\tilde X^1\tilde X^2=\tilde Y^1\tilde Y^2=0$ or, equivalently, $X^1Y^1=X^2Y^2=0$. Since $X^1Y^2-X^2Y^1\neq0$, the latter equations imply that either $X^1=Y^2=0$ or $X^2=Y^1=0$. It is obvious that the transformation $\mathcal{J}$: $\tilde t=t$, $\tilde x=y$, $\tilde y=x$, $\tilde u=u$, which just permutes x and y, is a point symmetry of the equation (1.1). Composing the corresponding point symmetries of the equation (1.1) with the transformation $\mathcal{J}$ reduces the case $X^1=Y^2=0$ to the case $X^2=Y^1=0$. Therefore, without loss of generality, we can assume in the rest of the proof that $X^2=Y^1=0$ and thus $X^1Y^2\neq0$. In the $\tilde u$-components pulled back by Φ, we can also collect the coefficients of x^2 and of y^2, which leads to the equations

$$U^1=(X^1)^2(\tilde\chi^{5x}_{\tilde t}(T)-t\tilde\chi^{4x}_{\tilde t}(T))=(Y^2)^2(\tilde\rho^{5x}_{\tilde t}(T)-t\tilde\rho^{4x}_{\tilde t}(T)).$$

Now we proceed with the equations from (1.5) with $i=4,5$ in the usual way. Considering $\tilde x$- and $\tilde y$-components, we derive the equations $\tilde\chi^{4x}(T)=X^1$, $\tilde\chi^{5x}(T)=tX^1$, $\tilde\rho^{4y}(T)=Y^2$, $\tilde\rho^{5y}(T)=tY^2$. Therefore, $(X^1)^3=(Y^2)^3=U^1T_t$, and thus

$$X^1=Y^2=F:=CT_t^{1/3}\neq0\quad\text{and}\quad U^1=C^3$$

with constant $C:=(U^1)^{1/3}\neq0$. The optimal way to obtain the rest of the equations of this step,

$$
\begin{aligned}
U^0_x = &\ -\frac{F_t}{2T_t}(Fx+X^0)^2+\tilde\alpha^{4x}(T)(Fx+X^0)\\
&+\tilde\beta^{4x}(T)(Fy+Y^0)+\tilde\sigma^{4x}(T),\\
U^0_y = &\ -\frac{F_t}{2T_t}(Fy+Y^0)^2+\tilde\alpha^{4y}(T)(Fx+X^0)\\
&+\tilde\beta^{4y}(T)(Fy+Y^0)+\tilde\sigma^{4y}(T),
\end{aligned}
$$

is to consider the $\tilde u$-components for $i=4$ and $z\in\{x,y\}$. The compatibility condition of these equations is $U^0_{xy}=U^0_{yx}$, giving $\tilde\beta^{4x}=\tilde\alpha^{4y}$. Their joint integration implies the representation

$$
U^0=-\frac{F^2F_t}{6T_t}(x^3+y^3)+W^3x^2+W^4xy+W^5y^2+W^1x+W^2y+W^0,
$$

where W^0, …, W^5 are smooth functions of t that are not constrained on this step.

There are two ways for further computations.

The first way is to implement the standard procedure for $i=6$, which gives the equations

$$
\begin{aligned}
&\lambda^6=1,\quad \tilde\chi^6(T)=-X^0,\quad \tilde\rho^6(T)=-Y^0,\\
&W^3-\frac12\tilde\chi^6_{\tilde t}(T)F^2=0,\quad W^4=0\quad\text{and}\quad W^5-\frac12\tilde\rho^6_{\tilde t}(T)F^2=0.
\end{aligned}
$$

Hence

$$
W^3=-\tfrac12C^2T_t^{-1/3}X^0_t,\quad W^5=-\tfrac12C^2T_t^{-1/3}Y^0_t,
$$

and we do not need to use the megaideal $\mathfrak m_1$.

The second way involves certain equations following from the equations with $i=7,8$ in (1.5). The corresponding computation is a bit more complicated than the above one and involves the megaideal $\mathfrak m_1$ instead of $\mathfrak m_2$. Nevertheless, as shown below in Section 1.3, the subalgebra of $\mathfrak g$ underlying this way has a nicer property than the analogous subalgebra for the first way.

Thus, up to composing with the transformation $\mathscr{J}$, the transformation Φ has the form (1.4) declared in the statement of the theorem. It is straightforward to check by the direct substitution that any point transformation of this form is a point symmetry of the equation (1.1).

(ii) To prove the equality $G_{\rm c} = G_{(1)}$, we apply the same modification of the megaideal-based method, where the maximal Lie invariance algebra $\mathfrak{g}$ is replaced with the contact invariance algebra $\mathfrak{g}_{\rm c} = \mathfrak{g}_{(1)}$, and the point transformation Φ is replaced with a contact transformation

$$\Psi\colon\ (\tilde t,\tilde x,\tilde y,\tilde u,\tilde u_{\tilde t},\tilde u_{\tilde x},\tilde u_{\tilde y})=(Z^t,Z^x,Z^y,U,U^t,U^x,U^y). \tag{1.8a}$$

In Ψ, the tuple on the right-hand side is a tuple of smooth functions of (t,x,y,u,u_t,u_x,u_y) with nonvanishing Jacobian, which additionally satisfies the contact condition

$$(Z^\mu_\nu+Z^\mu_u u_\nu)U^\mu=U_\nu+U_u u_\nu,\quad Z^\mu_{u_\nu}U^\mu=U_{u_\nu}. \tag{1.8b}$$

Here and in what follows the indices μ and ν run through the set $\{t,x,y\}$, and we assume summation for repeated indices. If the transformation Ψ is a contact symmetry of the equation (1.1), then $\Psi_*\mathfrak{m}_{j(1)}\subseteq\mathfrak{m}_{j(1)}$, $j=1,\dots,6$, where Ψ_* denotes the pushforward of contact vector fields by Ψ. To the counterpart of the collection of equations (1.5) for the contact case, we apply the procedure that is completely analogous to that described after (1.5). From the equation with $i=1$, we in particular derive the constraints $Z^\mu_u=0$. Then the equations with $i=2$, $(i,z)=(3,x)$ and $(i,z)=(3,y)$ imply the constraints $Z^\mu_{u_t}=0$, $Z^\mu_{u_x}=0$ and $Z^\mu_{u_y}=0$, respectively. In view of the contact condition, this means that $U_{u_\nu}=0$ as well, and thus the contact transformation Ψ is the first prolongation of a point transformation in the space with the coordinates (t,x,y,u). □

Each transformation Φ from the point-symmetry pseudogroup G of the dispersionless Nizhnik equation (1.1) can be represented as a composition of transformations from subgroups each of which is parameterized by a single functional or discrete parameter (below T, X^0, Y^0, W^0, W^1 and W^2 are arbitrary smooth functions of t with $T_t \neq 0$, and C is an arbitrary nonzero constant) in the following form:

$$
\begin{array}{llll}
\mathcal{D}^t(T)\colon & \tilde t = T, \ \ \tilde x = T_t^{1/3}x, & \tilde y = T_t^{1/3}y, & \\
& \tilde u = u - \frac{1}{18}T_{tt}T_t{}^{-1}(x^3+y^3), & & \\
\mathcal{D}^{\rm s}(C)\colon & \tilde t = t, \ \ \tilde x = Cx, & \tilde y = Cy, & \tilde u = C^3u, \\
\mathcal{P}^x(X^0)\colon & \tilde t = t, \ \ \tilde x = x + X^0, & \tilde y = y, & \\
& \tilde u = u - \frac16 X^0_t\big(3x^2+3X^0x+(X^0)^2\big), & & \\
\mathcal{P}^y(Y^0)\colon & \tilde t = t, \ \ \tilde x = x, & \tilde y = y + Y^0, & \\
& \tilde u = u - \frac16 Y^0_t\big(3y^2+3Y^0y+(Y^0)^2\big), & & \\
\mathcal{R}^x(W^1)\colon & \tilde t = t, \ \ \tilde x = x, & \tilde y = y, & \tilde u = u + W^1x, \\
\mathcal{R}^y(W^2)\colon & \tilde t = t, \ \ \tilde x = x, & \tilde y = y, & \tilde u = u + W^2y, \\
\mathcal{Z}(W^0)\colon & \tilde t = t, \ \ \tilde x = x, & \tilde y = y, & \tilde u = u + W^0, \\
\mathcal{J}\colon & \tilde t = t, \ \ \tilde x = y, & \tilde y = x, & \tilde u = u.
\end{array}
\tag{1.9}
$$

We will call transformations from the above families (1.9) elementary point symmetry transformations of the equation (1.1). Note that the subgroups $\{\mathcal{D}^t(T)\}$, $\{\mathcal{D}^{\rm s}(C)\}$, $\{\mathcal{P}^x(X^0)\}$, $\{\mathcal{P}^y(Y^0)\}$, $\{\mathcal{R}^x(W^1)\}$, $\{\mathcal{R}^y(W^2)\}$ and $\{\mathcal{Z}(W^0)\}$ of G are associated with the subalgebras $\{D^t(\tau)\}$, $\langle D^{\rm s}\rangle$, $\{P^x(\chi)\}$, $\{P^y(\rho)\}$, $\{R^x(\alpha)\}$, $\{R^y(\beta)\}$ and $\{Z(\sigma)\}$ of $\mathfrak{g}$, respectively. Here all the parameter functions run through the specified sets of their values. A representation of a transformation Φ of the form (1.4) as a composition of elementary point symmetry transformations of the equation (1.1) is

$$
\Phi = \mathcal{D}^t(T)\circ\mathcal{D}^{\rm s}(C)\circ\mathcal{P}^x(\tilde X^0)\circ\mathcal{P}^y(\tilde Y^0)\circ\mathcal{R}^x(\tilde W^1)\circ\mathcal{R}^y(\tilde W^2)\circ\mathcal{Z}(\tilde W^0)
$$

with

$$
\tilde X^0 = \frac{X^0}{CT_t^{1/3}}, \quad \tilde Y^0 = \frac{Y^0}{CT_t^{1/3}}, \quad \tilde W^0 = \frac{W^0}{C^3} + \frac{X^0_t(X^0)^2 + Y^0_t(Y^0)^2}{6C^3T_t},
$$
$$
\tilde W^1 = \frac{W^1}{C^3} + \frac{X^0_tX^0}{2C^2T_t^{2/3}}, \quad \tilde W^2 = \frac{W^2}{C^3} + \frac{Y^0_tY^0}{2C^2T_t^{2/3}}.
$$

Corollary 1.4. *The identity component $G_{\rm id}$ of the point-symmetry pseudogroup G of the dispersionless Nizhnik equation* (1.1) *consists of the transformations of the form* (1.4) *with $T_t>0$ and $C>0$. A complete list of discrete point symmetry transformations of the equation* (1.1) *that are independent up to composing with each other and with transformations from $G_{\rm id}$ is exhausted by three commuting involutions, which can be chosen to be the permutation $\mathcal J$ of the variables x and y, $(\tilde t,\tilde x,\tilde y,\tilde u)=(t,y,x,u)$, and two transformations $\mathcal I^{\rm i}$ and $\mathcal I^{\rm s}$ alternating the signs of (t,x,y) and of (x,y,u), respectively,*

$$\mathcal I^{\rm i}\colon\ (\tilde t,\tilde x,\tilde y,\tilde u)=(-t,-x,-y,u),\quad \mathcal I^{\rm s}\colon\ (\tilde t,\tilde x,\tilde y,\tilde u)=(t,-x,-y,-u).$$

Therefore, the quotient group $G/G_{\rm id}$ of the pseudogroup G with respect to its identity component $G_{\rm id}$ is isomorphic to the group $\mathbb Z_2\times\mathbb Z_2\times\mathbb Z_2$.

Remark 1.5. In Corollary 1.4 and analogous Corollaries 1.14 and 1.18 below, we assume that each of the listed discrete transformations is defined on the entire corresponding underlying space and absorbs its restrictions. The claims on the structure of the related discrete groups after the indicated corollaries are rigorous only under this assumption. The same assumption should be imposed on the elements of $\mathfrak H$ in the proof of Theorem 1.16 for $\mathfrak H$ to be a group.

1.3. Defining subalgebras for point transformations

In the course of applying the megaideal-based method in the proof of Theorem 1.3, we expand the basic condition $\Phi_*Q\in\mathfrak m$, where $\mathfrak m$ is the minimal megaideal of $\mathfrak g$ containing the vector field Q, only for 11 (linearly independent) vector fields from the algebra $\mathfrak g$, which is infinite-dimensional. These vector fields span an 11-dimensional subalgebra $\mathfrak s$ of $\mathfrak g$. In fact, we separately consider two subalgebras of $\mathfrak s$,

$$\mathfrak s_1=\langle Z(1),Z(t),R^x(1),R^y(1),P^x(1),P^y(1),P^x(t),P^y(t),D^{\rm s}\rangle,$$
$$\mathfrak s_2=\langle Z(1),Z(t),R^x(1),R^y(1),P^x(1),P^y(1),P^x(t),P^y(t),D^t(1),D^t(t)\rangle.$$

Moreover, in the other example of applying the same modification of the megaideal-based method in [85], which was computing the point-symmetry group of the Boiti–Leon–Pempinelli system, the selected linearly independent vector fields also span a subalgebra of the corresponding maximal Lie invariance algebra. Nevertheless, it is still not well understood how common this phenomenon is. That subalgebra has the following interesting property:

Definition 1.6. We call a proper subalgebra $\mathfrak{s}$ of a Lie algebra $\mathfrak{a}$ of vector fields a *subalgebra defining the diffeomorphisms that stabilize* $\mathfrak{a}$ if the conditions $\Phi_*\mathfrak{a}\subseteq\mathfrak{a}$ and $\Phi_*\mathfrak{s}\subseteq\mathfrak{a}$ for local diffeomorphisms Φ in the underlying space are equivalent.

The implication $\Phi_*\mathfrak{a}\subseteq\mathfrak{a}\Rightarrow\Phi_*\mathfrak{s}\subseteq\mathfrak{a}$ is obvious, whereas the inverse implication does not hold in general, and its verification requires nontrivial computations.

For a better understanding of the general foundations of the algebraic method in question, it is instructive to check whether the subalgebras $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are of the kind introduced in Definition 1.6.

Theorem 1.7. *The subalgebra $\mathfrak{s}_2$ of the algebra $\mathfrak{g}$ defines the diffeomorphisms that stabilize $\mathfrak{g}$, whereas the subalgebra $\mathfrak{s}_1$ and even the subalgebra $\bar{\mathfrak{s}}_1:=\mathfrak{s}_1+\langle D^t(1)\rangle$ does not have this property.*

Proof. We follow the proof of Theorem 1.3 and use the same numeration of the selected elements of the algebra $\mathfrak{g}$, but for each basis element Q of the subalgebra $\mathfrak{s}_1$ we employ the condition $\Phi_*Q\in\mathfrak{g}$ instead of the condition $\Phi_*Q\in\mathfrak{m}$, where $\mathfrak{m}$ is the minimal megaideal of $\mathfrak{g}$ containing the vector field Q. In other words, we replace the equations (1.5) with the equations

$$\begin{aligned}\Phi_*Q^\kappa=\lambda^\kappa\tilde D^{\rm s}+\tilde D^t(\tilde\tau^\kappa)+\tilde P^x(\tilde\chi^\kappa)+\tilde P^y(\tilde\rho^\kappa)\\ +\tilde R^x(\tilde\alpha^\kappa)+\tilde R^y(\tilde\beta^\kappa)+\tilde Z(\tilde\sigma^\kappa),\end{aligned}\tag{1.10}$$

where λ^κ are constants, $\tilde\tau^\kappa$, $\tilde\chi^\kappa$, $\tilde\rho^\kappa$, $\tilde\alpha^\kappa$, $\tilde\beta^\kappa$ and $\tilde\sigma^\kappa$ are smooth functions of $\tilde t$, and the index κ runs the set $\big\{1,2,3z,4z,5z,6,7,8\mid z\in\{x,y\}\big\}$.

Collecting $\tilde t$-components in the equations with $\kappa = 1, 2$, we derive the equations $T_u = \tilde\tau^1(T)$ and $tT_u = \tilde\tau^2(T)$. Suppose that $T_u \neq 0$, and thus $\tilde\tau^1\tilde\tau^2 \neq 0$. Recombining the above equations leads to the equation $t = f(T)$ with $f(\tilde t) := \tilde\tau^2(\tilde t)/\tilde\tau^1(\tilde t)$. Differentiating it with respect to t gives $1 = f_{\tilde t}(T)T_t$. Therefore, the derivative $f_{\tilde t}$ does not vanish, and according to the inverse function theorem, we obtain that $T = T(t)$, which contradicts the supposition $T_u \neq 0$. Therefore, $T_u = 0$ and also $\tilde\tau^1 = \tilde\tau^2 = 0$.

Collecting $\tilde x$- and $\tilde y$-components in the same equations with $\kappa = 1, 2$ leads to the equations $X_u = \lambda^1 X + \tilde\chi^1(T)$, $tX_u = \lambda^2 X + \tilde\chi^2(T)$, $Y_u = \lambda^1 Y + \tilde\rho^1(T)$ and $tY_u = \lambda^2 Y + \tilde\rho^2(T)$, which can be combined to $(\lambda^1 t - \lambda^2)X + t\tilde\chi^1(T) - \tilde\chi^2(T) = 0$ and $(\lambda^1 t - \lambda^2)Y + t\tilde\rho^1(T) - \tilde\rho^2(T) = 0$. Suppose that $(X_u, Y_u) \neq (0, 0)$. Then we can split at least one of the last two equations with respect to X or Y, respectively. As a result, we obtain the equation $\lambda^1 t - \lambda^2 = 0$, which splits further with respect to t to $\lambda^1 = \lambda^2 = 0$. Therefore, we also have $t\tilde\chi^1(T) = \tilde\chi^2(T)$ and $t\tilde\rho^1(T) = \tilde\rho^2(T)$. Moreover, $(\tilde\chi^1\tilde\chi^2, \tilde\rho^1\tilde\rho^2) \neq (0, 0)$ due to the supposition $(X_u, Y_u) \neq (0, 0)$. Following the consideration of t-components, we again derive an equation of the form $t = f(T)$ with $f_{\tilde t} \neq 0$ and obtain in view of the inverse function theorem that $T = T(t)$. Then we collect $\tilde u$-components in the same equations with $\kappa = 1, 2$ and derive the equations

$$
\begin{aligned}
U_u &= -\frac12\tilde\chi^1_{\tilde t}(T)X^2 - \frac12\tilde\rho^1_{\tilde t}(T)Y^2 + \tilde\alpha^1(T)X + \tilde\beta^1(T)Y + \tilde\sigma^1(T),\\
tU_u &= -\frac12\tilde\chi^2_{\tilde t}(T)X^2 - \frac12\tilde\rho^2_{\tilde t}(T)Y^2 + \tilde\alpha^2(T)X + \tilde\beta^2(T)Y + \tilde\sigma^2(T).
\end{aligned}
$$

We subtract the second equation from the first one multiplied by t. Since t, X and Y are functionally independent, the equation obtained in this way can be split with respect to (X, Y), which in particular results in the equations $t\tilde\chi^1_{\tilde t}(T) = \tilde\chi^2_{\tilde t}(T)$, $t\tilde\rho^1_{\tilde t}(T) = \tilde\rho^2_{\tilde t}(T)$. Pairwise differential consequences of these equations jointly with the equations $t\tilde\chi^1(T) = \tilde\chi^2(T)$ and $t\tilde\rho^1(T) = \tilde\rho^2(T)$ are the equations $\tilde\chi^1 = \tilde\chi^2 = 0$ and $\tilde\rho^1 = \tilde\rho^2 = 0$, respectively. Therefore, we have the equations $X_u = Y_u = 0$, which contradict the supposition $(X_u, Y_u) \neq (0,0)$. This is why in fact $X_u = Y_u = 0$ as well as $\lambda^1 = \lambda^2 = 0$, $\tilde\chi^1 = \tilde\chi^2 = 0$, $\tilde\rho^1 = \tilde\rho^2 = 0$ and $U_u \neq 0$.

Under the derived constraints, the only essential equation that is obtained via collecting $\tilde u$-components in the equations with $\kappa = 1, 2$ is

$$U_u = \tilde\alpha^1(T)X + \tilde\beta^1(T)Y + \tilde\sigma^1(T).$$

We temporarily jump to the equations with $\kappa = 4z, 5z$, $z \in \{x, y\}$, where we only collect $\tilde t$-components on this step, obtaining $T_z = \tilde\tau^{4z}(T)$, $tT_z = \tilde\tau^{5z}(T)$, and thus $t\tilde\tau^{4z}(T) = \tilde\tau^{5z}(T)$. Supposing that $T_z \neq 0$ for some $z \in \{x, y\}$, we then have $\tilde\tau^{4z}(T) \neq 0$ and $t = f(T)$ with $f = \tilde\tau^{5z}/\tilde\tau^{4z}$. Using the same arguments as at the beginning of the proof, we obtain that the function T depends only on t, which contradicts the supposition $T_z \neq 0$. Hence $T_x = T_y = 0$, i.e., T is nevertheless a function of t only, $T = T(t)$ with $T_t \neq 0$, and also $\tilde\tau^{4z} = \tilde\tau^{5z} = 0$.

We return to the equations with $\kappa = 3z$, $z \in \{x, y\}$, which we also consider simultaneously. We successively collect $\tilde t$-, $\tilde x$- and $\tilde y$-components and split the obtained equations with respect to X and Y since the functions T, X and Y are functionally independent. This gives the constraints $\tilde\tau^{3z} = \tilde\chi^{3z} = \tilde\rho^{3z} = 0$, $\lambda^{3z} = 0$. Then collecting of $\tilde u$-components leads to the constraints $zU_u = \tilde\alpha^{3z}(T)X + \tilde\beta^{3z}(T)Y + \tilde\sigma^{3z}(T)$. In view of the above expression for U_u, this means that x and y can be represented as linear fractional functions of (X, Y) with coefficients depending on T. Since the inverse Φ^{-1} belongs to the pseudogroup G as the transformation Φ does, we can permit (t, x, y) and (T, X, Y) in the last claim. In other words, X and Y are linear fractional functions of (x, y) with coefficients depending on t, $X = \mathrm{N}^X/\mathrm{D}$ and $Y = \mathrm{N}^Y/\mathrm{D}$, where the numerators and the denominator respectively are

$$\mathrm{N}^X = X^1(t)x + X^2(t)y + X^0(t), \quad \mathrm{N}^Y = Y^1(t)x + Y^2(t)y + Y^0(t),$$
$$\mathrm{D} = K^1(t)x + K^2(t)y + K^0(t)$$

for some smooth functions X^0, X^1, X^2, Y^0, Y^1, Y^2, K^0, K^1 and K^2 of t.

Now we consider the equations with $\kappa = 4z$, $z \in \{x, y\}$. The equations $X_z = \lambda^{4z}X + \tilde\chi^{4z}(T)$ and $Y_z = \lambda^{4z}Y + \tilde\rho^{4z}(T)$ which are obtained by successively collecting $\tilde x$- and $\tilde y$-components, reduce to

$$
\begin{aligned}
&X^1\mathrm{D} - K^1\mathrm{N}^X = \lambda^{4x}\mathrm{N}^X\mathrm{D} + \tilde\chi^{4x}(T)\mathrm{D}^2,\\
&X^2\mathrm{D} - K^2\mathrm{N}^X = \lambda^{4y}\mathrm{N}^X\mathrm{D} + \tilde\chi^{4y}(T)\mathrm{D}^2,\\
&Y^1\mathrm{D} - K^1\mathrm{N}^Y = \lambda^{4x}\mathrm{N}^Y\mathrm{D} + \tilde\rho^{4x}(T)\mathrm{D}^2,\\
&Y^2\mathrm{D} - K^2\mathrm{N}^Y = \lambda^{4y}\mathrm{N}^Y\mathrm{D} + \tilde\rho^{4y}(T)\mathrm{D}^2.
\end{aligned}
\tag{1.11}
$$

Suppose that $(K^1, K^2) \neq (0, 0)$. Then $X^1K^2 - X^2K^1 \neq 0$ or $Y^1K^2 - Y^2K^1 \neq 0$ since otherwise the Jacobian of the functions T, X and Y is zero. Recall that the point transformation $\mathscr{J}$: $\tilde t = t$, $\tilde x = y$, $\tilde y = x$, $\tilde u = u$, which just permutes x and y, is an obvious point symmetry of the equation (1.1). This is why we can assume without loss of generality that $X^1K^2 - X^2K^1 \neq 0$. Hence the Jacobian of the functions N^X and D with respect to (x, y) is nonzero, and we can split the first two equations in (1.11) with respect to $(\mathrm{N}^X, \mathrm{D})$. As a result, we in particular derive the constraints $X^1 = X^2 = 0$, which contradict the inequality $X^1K^2 - X^2K^1 \neq 0$. Therefore, $K^1 = K^2 = 0$, i.e., the functions X and Y are affine in (x, y) with coefficients depending on t. Re-denoting X^k/K^0 by X^k and Y^k/K^0 by Y^k, $k = 0, 1, 2$, we can completely follow the part with $i = 4, 5$ in item (i) of the proof of Theorem 1.3.

The computation for $\kappa = 6$ and further is again different.

Taking only the $\tilde t$-components and the coefficients of x in the $\tilde x$-components or, equivalently, the coefficients of y in the $\tilde y$-components in (1.10) with $\kappa = 7$ and with $\kappa = 8$, we respectively obtain

$$
\begin{aligned}
&T_t = \tilde\tau^7(T), \quad F_t/F = \frac{1}{3}T_{tt}/T_t + \lambda^7,\\
&tT_t = \tilde\tau^8(T), \quad tF_t/F = \frac{1}{3}tT_{tt}/T_t + \lambda^8.
\end{aligned}
$$

Combining the second and fourth equations to exclude F_t/F gives $t\lambda^7 = \lambda^8$, i.e., $\lambda^7 = \lambda^8 = 0$, and thus we can complete the proof for the subalgebra $\mathfrak{s}_2$

as in the second way in item (i) on the proof of Theorem 1.3. Therefore, the condition $\Phi_*\mathfrak{s}_2 \subseteq \mathfrak{g}$ implies $\Phi_*\mathfrak{g} \subseteq \mathfrak{g}$. In other words, the subalgebra $\mathfrak{s}_2$ defines the diffeomorphisms that stabilize $\mathfrak{g}$.

If we use the equations (1.10) with $\kappa = 6, 7$ instead of those with $\kappa = 7, 8$, then we again obtain the equations

$$\frac{F_t}{F} = \frac{T_{tt}}{3T_t} + \lambda^7, \quad W^3 = -\frac{F^2}{2T_t}X^0_t, \quad W^4 = 0, \quad W^5 = -\frac{F^2}{2T_t}Y^0_t$$

for transformation parameters, and only these equations and their differential consequences. Here the parameter λ^7 is an arbitrary constant, and thus the set of point transformations Φ satisfying the condition $\Phi_*\mathfrak{s}_1 \subseteq \mathfrak{g}$ properly contains the group G, which coincides, in view of Theorem 1.3, with the set of point transformations Φ satisfying the condition $\Phi_*\mathfrak{g} \subseteq \mathfrak{g}$. Therefore, the subalgebra $\bar{\mathfrak{s}}_1$ does not define completely the diffeomorphisms that stabilize $\mathfrak{g}$. Then the subalgebra $\mathfrak{s}_1$ as that contained in $\bar{\mathfrak{s}}_1$ all the more has the same property. □

1.4. Defining subalgebras for contact transformations

Subalgebras defining the diffeomorphisms that stabilize the entire corresponding algebras can also be considered for algebras of contact vector fields and local contact diffeomorphisms. The transition from the point case to the contact one complicates the problem due to extending the space coordinatized by the independent and dependent variables with the first-order jet variables and thus essentially increasing the total number of coordinates.

Theorem 1.7 implies that the first prolongation $\mathfrak{s}_{1(1)}$ of the subalgebra $\mathfrak{s}_1$ of $\mathfrak{g}$ does not define local contact diffeomorphisms that stabilize $\mathfrak{g}_{(1)}$ since the subalgebra $\mathfrak{s}_1$ itself does not define local diffeomorphisms that stabilize $\mathfrak{g}$. It is not clear whether the first prolongation $\mathfrak{s}_{2(1)}$ of the subalgebra $\mathfrak{s}_2$ of $\mathfrak{g}$ differs from $\mathfrak{s}_{1(1)}$ in the sense of defining local contact diffeomorphisms that stabilize $\mathfrak{g}_{(1)}$. To answer this question it is necessary to integrate cumbersome parameterized nonlinear overdetermined systems of differential equations, and its solution requires more sophisticated techniques than those used in the proofs of Theorems 1.3 and 1.7. The latter techniques are still efficient only if we extend the algebra to be tested.

Theorem 1.8. *A contact transformation Ψ with the basic space $\mathbb{R}^3_{t,x,y}\times\mathbb{R}_u$ satisfies the condition $\Psi_*\mathfrak{s}_{3(1)}\subseteq\mathfrak{g}_{(1)}$ for the subalgebra*

$$\mathfrak{s}_3=\langle Z(1),Z(t),Z(t^2),R^x(1),R^y(1),R^x(t),R^y(t)\rangle$$

of the algebra $\mathfrak{g}$ only if it is the first prolongation of a point transformation in the above space.

Proof. Suppose that a contact transformation Ψ with the basic space $\mathbb{R}^3_{t,x,y}\times\mathbb{R}_u$, which is of the general form (1.8), satisfies the condition $\Psi_*\mathfrak{s}_{3(1)}\subseteq\mathfrak{g}_{(1)}$. For convenience, we re-denote the basis elements of $\mathfrak{s}_3$ as

$$\mathcal{Q}^1:=Z(1),\quad \mathcal{Q}^2:=Z(t),\quad \mathcal{Q}^3:=Z(t^2),$$
$$\mathcal{Q}^4:=R^x(1),\quad \mathcal{Q}^5:=R^y(1),\quad \mathcal{Q}^6:=R^x(t),\quad \mathcal{Q}^7:=R^y(t).$$

Then we expand the condition $\Psi_*\mathfrak{s}_{3(1)}\subseteq\mathfrak{g}_{(1)}$ to

$$\begin{aligned}\Psi_*\mathcal{Q}^i_{(1)}=\lambda^i\tilde D^{\rm s}_{(1)}+\tilde D^t_{(1)}(\tilde\tau^i)+\tilde P^x_{(1)}(\tilde\chi^i)+\tilde P^y_{(1)}(\tilde\rho^i)\\ +\tilde R^x_{(1)}(\tilde\alpha^i)+\tilde R^y_{(1)}(\tilde\beta^i)+\tilde Z_{(1)}(\tilde\sigma^i),\end{aligned}\tag{1.12}$$

where λ^i are constants, $\tilde\tau^i$, $\tilde\chi^i$, $\tilde\rho^i$, $\tilde\alpha^i$, $\tilde\beta^i$ and $\tilde\sigma^i$ are smooth functions of $\tilde t$, and the index i runs from 1 to 7.

Collecting t-components in the equations (1.12) with $i=1,2,3$, we derive the equations

$$T_u=\tilde\tau^1(T),\quad tT_u+T_{u_t}=\tilde\tau^2(T),\quad t^2T_u+2tT_{u_t}=\tilde\tau^3(T),$$

whose algebraic consequence is the equation $\tilde\tau^1(T)t^2-2\tilde\tau^2(T)t+\tilde\tau^3(T)=0$. Suppose that $(\tilde\tau^1,\tilde\tau^2)\neq(0,0)$. Then the last equation implies that

$t = f(T)$, and, similarly to the proof of Theorem 1.7, we successively have $T = T(t)$, $\tilde\tau^1 = 0$ and $\tilde\tau^2 = 0$, which contradicts the supposition. Hence $\tilde\tau^1 = \tilde\tau^2 = 0$ and $T_u = T_{u_t} = 0$, and thus $\tilde\tau^3 = 0$ as well.

The next step is to collect x- and y-components in the same equations with $i = 1, 2, 3$. It results in the equations

$$\begin{aligned}
&X_u = \lambda^1 X + \tilde\chi^1(T), \quad tX_u + X_{u_t} = \lambda^2 X + \tilde\chi^2(T),\\
&t^2 X_u + 2tX_{u_t} = \lambda^3 X + \tilde\chi^3(T),\\
&Y_u = \lambda^1 Y + \tilde\rho^1(T), \quad tY_u + Y_{u_t} = \lambda^2 Y + \tilde\rho^2(T),\\
&t^2 Y_u + 2tY_{u_t} = \lambda^3 Y + \tilde\rho^3(T).
\end{aligned}$$

We separately combine the equations in each row to exclude derivatives of X and Y,

$$\begin{aligned}
&(t^2\lambda^1 + 2t\lambda^2 - \lambda^3)X + t^2\tilde\chi^1(T) + 2t\tilde\chi^2(T) - \tilde\chi^3(T) = 0,\\
&(t^2\lambda^1 + 2t\lambda^2 - \lambda^3)Y + t^2\tilde\rho^1(T) + 2t\tilde\rho^2(T) - \tilde\rho^3(T) = 0.
\end{aligned}$$

Suppose that $(X_u, X_{u_t}, Y_u, Y_{u_t}) \neq (0, 0, 0, 0)$. Then, we can split the last system with respect to X and Y, which leads to the equations $t^2\lambda^1 + 2t\lambda^2 - \lambda^3 = 0$,

$$\begin{aligned}
&t^2\tilde\chi^1(T) + 2t\tilde\chi^2(T) - \tilde\chi^3(T) = 0,\\
&t^2\tilde\rho^1(T) + 2t\tilde\rho^2(T) - \tilde\rho^3(T) = 0.
\end{aligned} \tag{1.13a}$$

The first equation means that $\lambda^1 = \lambda^2 = \lambda^3 = 0$, and hence $(\tilde\chi^1, \tilde\chi^2, \tilde\rho^1, \tilde\rho^2) \neq (0, 0, 0, 0)$. Following the above consideration of t-components, we obtain that the function T depends only on t, $T = T(t)$. We continue the analysis of the equations (1.12) with $i = 1, 2, 3$, collecting u-components. This gives the equations

$$\begin{aligned}
&U_u = -\frac{1}{2}\tilde\chi^1_{\tilde t}(T)X^2 - \frac{1}{2}\tilde\rho^1_{\tilde t}(T)Y^2 + \tilde\alpha^1(T)X + \tilde\beta^1(T)Y + \tilde\sigma^1(T),\\
&tU_u + U_{u_t} = -\frac{1}{2}\tilde\chi^2_{\tilde t}(T)X^2 - \frac{1}{2}\tilde\rho^2_{\tilde t}(T)Y^2 + \tilde\alpha^2(T)X + \tilde\beta^2(T)Y + \tilde\sigma^2(T),\\
&t^2U_u + 2tU_{u_t} = -\frac{1}{2}\tilde\chi^3_{\tilde t}(T)X^2 - \frac{1}{2}\tilde\rho^3_{\tilde t}(T)Y^2 + \tilde\alpha^3(T)X + \tilde\beta^3(T)Y + \tilde\sigma^3(T).
\end{aligned}$$

We linearly combine the first, the second and the third equations with coefficients t^2, $-2t$ and 1, respectively. We can split the obtained algebraic consequence of these equations with respect to X and Y since T, X and Y are functionally independent, T depends on t only and thus t, X and Y are functionally independent. As a result, we derive the equations

$$\begin{aligned} &t^2\tilde\chi^1_{\tilde t}(T)+2t\tilde\chi^2_{\tilde t}(T)-\tilde\chi^3_{\tilde t}(T)=0,\\ &t^2\tilde\rho^1_{\tilde t}(T)+2t\tilde\rho^2_{\tilde t}(T)-\tilde\rho^3_{\tilde t}(T)=0. \end{aligned}\tag{1.13b}$$

In view of them, the analogous consideration of u_t-components in the equations with $i=1,2,3$ gives the equations

$$\begin{aligned} &t^2\tilde\chi^1_{\tilde t\tilde t}(T)+2t\tilde\chi^2_{\tilde t\tilde t}(T)-\tilde\chi^3_{\tilde t\tilde t}(T)=0,\\ &t^2\tilde\rho^1_{\tilde t\tilde t}(T)+2t\tilde\rho^2_{\tilde t\tilde t}(T)-\tilde\rho^3_{\tilde t\tilde t}(T)=0. \end{aligned}\tag{1.13c}$$

We construct, separately for the equations with respect to χ and for the equations with respect to ρ, the differential consequences of the system (1.13) that have the structures

$$\partial_t(1.13\text{a})-T_t(1.13\text{b}),\quad \partial_t^2(1.13\text{a})-(2T_t\partial_t+T_{tt})(1.13\text{b})+{T_t}^2(1.13\text{c}).$$

After the additional division by 2, these differential consequences take the form

$$t\tilde\chi^1(T)+\tilde\chi^2(T)=0,\quad t\tilde\rho^1(T)+\tilde\rho^2(T)=0,\quad \tilde\chi^1(T)=0,\quad \tilde\rho^1(T)=0$$

and this implies, jointly with (1.13a) that $\tilde\chi^1=\tilde\chi^2=\tilde\chi^3=0$ and $\tilde\rho^1=\tilde\rho^2=\tilde\rho^3=0$, which contradicts the supposition $(X_u,X_{u_t},Y_u,Y_{u_t})\neq(0,0,0,0)$. Hence $X_u=Y_u=X_{u_t}=Y_{u_t}=0$.

The next step is to collect t-components in the equations with $i=4,5$. It gives the equations $T_{u_z}=\tilde\tau^{4z}(T)$ and $tT_{u_z}=\tilde\tau^{5z}(T)$ with $z\in\{x,y\}$. Suppose that $T_{u_z}\neq0$. Then, similarly to the beginning of the proof of Theorem 1.7 we again derive that $T=T(t)$ and thus $T_{u_z}=0$, which contradicts the supposition. This implies $T_{u_z}=0$ and $\tilde\tau^{4z}=\tilde\tau^{5z}=0$ as well.

Collecting x- and y-components in the same equations with $i = 4, 5$ leads to the equations

$$X_{u_z} = \lambda^{4z} X + \tilde{\chi}^{4z}(T), \quad tX_{u_z} = \lambda^{5z} X + \tilde{\chi}^{5z}(T),$$
$$Y_{u_z} = \lambda^{4z} Y + \tilde{\rho}^{4z}(T), \quad tY_{u_z} = \lambda^{5z} Y + \tilde{\rho}^{5z}(T).$$

They are combined to $(t\lambda^{4z} - \lambda^{5z})X + t\tilde{\chi}^{4z} - \tilde{\chi}^{5z} = 0$ and $(t\lambda^{4z} - \lambda^{5z})Y + t\tilde{\rho}^{4z} - \tilde{\rho}^{5z} = 0$. Suppose that $(X_{u_x}, Y_{u_x}, X_{u_y}, Y_{u_y}) \neq (0, 0, 0, 0)$. Then we can successively split these combinations with respect to X and Y and in addition split the coefficients of X and Y with respect to t, which gives $\lambda^{4z} = \lambda^{5z} = 0$, $t\tilde{\chi}^{4z} = \tilde{\chi}^{5z}$ and $t\tilde{\rho}^{4z} = \tilde{\rho}^{5z}$ with $z \in \{x, y\}$. After collecting u-components in the equations with $i = 4, 5$, we have

$$zU_u + U_{u_z} = -\frac{1}{2}\tilde{\chi}^{4z}_{\tilde{t}}(T)X^2 - \frac{1}{2}\tilde{\rho}^{4z}_{\tilde{t}}(T)Y^2 + \tilde{\alpha}^{4z}(T)X + \tilde{\beta}^{4z}(T)Y + \tilde{\sigma}^{4z}(T),$$
$$t(zU_u + U_{u_z}) = -\frac{1}{2}\tilde{\chi}^{5z}_{\tilde{t}}(T)X^2 - \frac{1}{2}\tilde{\rho}^{5z}_{\tilde{t}}(T)Y^2 + \tilde{\alpha}^{5z}(T)X + \tilde{\beta}^{5z}(T)Y + \tilde{\sigma}^{5z}(T).$$

We combine the last equations, subtracting the first equation multiplied by t from the second one. Splitting the combination with respect to X and Y gives $t\tilde{\chi}^{4z}_{\tilde{t}} - \tilde{\chi}^{5z}_{\tilde{t}} = 0$ and $t\tilde{\rho}^{4z}_{\tilde{t}} - \tilde{\rho}^{5z}_{\tilde{t}} = 0$. Differential consequences of the derived system for χ^{4z} and ρ^{5z} are the equations $\tilde{\chi}^{4z} = \tilde{\chi}^{5z} = \tilde{\rho}^{4z} = \tilde{\rho}^{5z} = 0$, which obviously imply $X_{u_x} = Y_{u_x} = X_{u_y} = Y_{u_y} = 0$. The obtained contradiction with the supposition $(X_{u_x}, Y_{u_x}, X_{u_y}, Y_{u_y}) \neq (0, 0, 0, 0)$ means that we ultimately have $X_{u_x} = Y_{u_x} = X_{u_y} = Y_{u_y} = 0$.

Due to the independence of (T, X, Y) on u_t, u_x, u_y, it follows from the contact condition (1.8b) that $U_{u_t} = U_{u_x} = U_{u_y} = 0$ as well. Therefore, the contact transformation Ψ is the first prolongation of a point transformation in the basic space $\mathbb{R}^3_{t,x,y} \times \mathbb{R}_u$. □

Corollary 1.9. *The first prolongation of the subalgebra $\mathfrak{s}_2 + \mathfrak{s}_3$, which is a subalgebra of the algebra $\mathfrak{g}_{\rm c} = \mathfrak{g}_{(1)}$, defines the diffeomorphisms of the corresponding first-order jet space that stabilize $\mathfrak{g}_{\rm c}$.*

Proof. If a contact transformation Ψ with the basic space $\mathbb{R}^3_{t,x,y}\times\mathbb{R}_u$ satisfies the condition $\Psi_*(\mathfrak{s}_2+\mathfrak{s}_3)_{(1)}\subseteq\mathfrak{g}_{(1)}$, then it satisfies the weaker condition $\Psi_*\mathfrak{s}_{3(1)}\subseteq\mathfrak{g}_{(1)}$. In view of Theorem 1.8, this implies that the contact transformation Ψ is the first prolongation of a point transformation Φ in the basic space $\mathbb{R}^3_{t,x,y}\times\mathbb{R}_u$, $\Psi=\Phi_{(1)}$ and the condition $\Psi_*(\mathfrak{s}_2+\mathfrak{s}_3)_{(1)}\subseteq\mathfrak{g}_{(1)}$ reduces to the condition $\Phi_*(\mathfrak{s}_2+\mathfrak{s}_3)\subseteq\mathfrak{g}$. In particular, $\Phi_*\mathfrak{s}_2\subseteq\mathfrak{g}$. According to Theorem 1.7, we have $\Phi_*\mathfrak{g}\subseteq\mathfrak{g}$. The first prolongation of the last condition gives the required property of Ψ, $\Psi_*\mathfrak{g}_{(1)}\subseteq\mathfrak{g}_{(1)}$. □

1.5. Point-symmetry pseudogroup of nonlinear Lax representation

A nonlinear Lax representation of the dispersionless Nizhnik equation (1.1),

$$v_t=\frac13\left(v_x^3-\frac{u_{xy}^3}{v_x^3}\right)+u_{xx}v_x-\frac{u_{xy}u_{yy}}{v_x},\quad v_y=-\frac{u_{xy}}{v_x},\tag{1.14}$$

was derived as a dispersionless counterpart[1.3] of the Lax representation of the Nizhnik equation, cf. [104]. The maximal Lie invariance (pseudo)algebra $\mathfrak{g}_{\rm L}$ of the system (1.14) is spanned by the vector fields

$$\begin{aligned}
&\bar D^t(\tau)=\tau\partial_t+\tfrac13\tau_tx\partial_x+\tfrac13\tau_ty\partial_y-\tfrac1{18}\tau_{tt}(x^3+y^3)\partial_u,\\
&\bar D^{\rm s}=x\partial_x+y\partial_y+3u\partial_u+\tfrac32v\partial_v,\\
&\bar P^x(\chi)=\chi\partial_x-\tfrac12\chi_tx^2\partial_u,\quad \bar P^y(\rho)=\rho\partial_y-\tfrac12\rho_ty^2\partial_u,\\
&\bar R^x(\alpha)=\alpha x\partial_u,\quad \bar R^y(\beta)=\beta y\partial_u,\quad \bar Z(\sigma)=\sigma\partial_u,\quad \bar P^v=\partial_v,
\end{aligned}$$

where τ, χ, ρ, α, β and σ are again arbitrary smooth functions of t. The algebra $\mathfrak{g}_{\rm L}$ is infinite-dimensional as the algebra $\mathfrak{g}$ and is obtained from $\mathfrak{g}$ by extending the vector fields from $\mathfrak{g}$ to the additional dependent variable v and supplementing the extended algebra with the vector field $\bar P^v$. The

[1.3] See a technique of limit transitions to dispersionless counterparts of (1+2)-dimensional differential equations and of the corresponding Lax representations in [131, p. 167].

appearance of $\bar{P}^v$ is natural and related to the fact that the unknown function v is defined up to a constant summand. This is why we can say that the maximal Lie invariance algebra $\mathfrak{g}_{\rm L}$ of the system (1.14) is induced by the maximal Lie invariance algebra $\mathfrak{g}$ of the equation (1.1). Up to the antisymmetry of the Lie bracket, the nonzero commutation relations between vector fields spanning $\mathfrak{g}_{\rm L}$ are exhausted by the counterparts of the commutation relations (1.3) and one more commutation relation involving the vector field $\bar{P}^v$, $[\bar{P}^v, \bar{D}^{\rm s}] = \frac{3}{2}\bar{P}^v$.

Remark 1.10. The corresponding linear nonisospectral Lax representation has the following form:

$$
\begin{aligned}
\chi_t &= (p^2 + p^{-4}u_{xy}^3 + u_{xx} + p^{-2}u_{xy}u_{yy})\chi_x \\
&\quad - (pu_{xxx} - p^{-1}(u_{xy}u_{yy})_x - p^{-3}u_{xy}^2u_{xxy})\chi_p, \\
\chi_y &= p^{-2}u_{xy}\chi_x + p^{-1}u_{xxy}\chi_p,
\end{aligned}
$$

where p is a variable spectral parameter, $\chi = \chi(t, x, y, p)$ and $u = u(t, x, y)$. See, for example, [122, p. 360] and references therein for linear nonisospectral Lax representations and the procedure of converting a nonlinear Lax representation into its linear nonisospectral counterpart in the (1+2)-dimensional case.

Analogously to Lemma 1.2, we can prove the following assertion.

Lemma 1.11. *The radical of* $\mathfrak{g}_{\rm L}$ *is*

$$
\mathfrak{r}_{\rm L} = \big\langle \bar{D}^{\rm s}, \bar{P}^x(\chi), \bar{P}^y(\rho), \bar{R}^x(\alpha), \bar{R}^y(\beta), \bar{Z}(\sigma), \bar{P}^v \big\rangle.
$$

Further following the consideration of Section 1.1, we construct several megaideals of the algebra $\mathfrak{g}_{\rm L}$,

$$
\begin{aligned}
&\mathfrak{g}_{\rm L}{}' = \big\langle \bar{D}^t(\tau), \bar{P}^x(\chi), \bar{P}^y(\rho), \bar{R}^x(\alpha), \bar{R}^y(\beta), \bar{Z}(\sigma), \bar{P}^v \big\rangle, \\
&\bar{\mathfrak{m}}_1 := \mathfrak{g}_{\rm L}{}'' = \big\langle \bar{D}^t(\tau), \bar{P}^x(\chi), \bar{P}^y(\rho), \bar{R}^x(\alpha), \bar{R}^y(\beta), \bar{Z}(\sigma) \big\rangle, \\
&\bar{\mathfrak{m}}_2 := \mathfrak{r}_{\rm L} = \big\langle \bar{D}^{\rm s}, \bar{P}^x(\chi), \bar{P}^y(\rho), \bar{R}^x(\alpha), \bar{R}^y(\beta), \bar{Z}(\sigma), \bar{P}^v \big\rangle, \\
&\bar{\mathfrak{m}}_2' = \mathfrak{g}_{\rm L}{}' \cap \bar{\mathfrak{m}}_2 = \big\langle \bar{P}^x(\chi), \bar{P}^y(\rho), \bar{R}^x(\alpha), \bar{R}^y(\beta), \bar{Z}(\sigma), \bar{P}^v \big\rangle,
\end{aligned}
$$

$$\bar{\mathfrak{m}}_3 := \bar{\mathfrak{m}}_1 \cap \bar{\mathfrak{m}}_2' = \big\langle \bar{P}^x(\chi), \bar{P}^y(\rho), \bar{R}^x(\alpha), \bar{R}^y(\beta), \bar{Z}(\sigma)\big\rangle,$$
$$\bar{\mathfrak{m}}_4 := \bar{\mathfrak{m}}_2'' = \big\langle \bar{R}^x(\alpha), \bar{R}^y(\beta), \bar{Z}(\sigma)\big\rangle, \quad \bar{\mathfrak{m}}_5 := \bar{\mathfrak{m}}_2''' = \big\{\bar{Z}(\sigma)\big\},$$
$$\mathfrak{z}(\mathfrak{g}_{\rm L}{}') = \big\langle Z(1), \bar{P}^v\big\rangle, \quad \bar{\mathfrak{m}}_6 := \mathfrak{z}(\mathfrak{g}_{\rm L}{}'') = \big\langle Z(1)\big\rangle, \quad \bar{\mathfrak{m}}_7 := \big\langle \bar{P}^v\big\rangle.$$

The technique for finding the megaideal $\bar{\mathfrak{m}}_7$ differs from that for the other obtained megaideals. This is why we formulate the claim on $\bar{\mathfrak{m}}_7$ as an assertion.

Lemma 1.12. *The span* $\bar{\mathfrak{m}}_7 := \big\langle \bar{P}^v\big\rangle$ *is a megaideal of* $\mathfrak{g}_{\rm L}$.

Proof. Let φ be an arbitrary automorphism of $\mathfrak{g}_{\rm L}$. Since $\bar{D}^{\rm s} \in \bar{\mathfrak{m}}_2 \setminus \bar{\mathfrak{m}}_2'$ and $\bar{P}^v \in \mathfrak{z}(\mathfrak{g}_{\rm L}{}') \setminus \bar{\mathfrak{m}}_6$, where $\bar{\mathfrak{m}}_2 := \mathfrak{r}_{\rm L}$, $\bar{\mathfrak{m}}_2'$, $\mathfrak{z}(\mathfrak{g}_{\rm L}{}')$ and $\bar{\mathfrak{m}}_6 := \mathfrak{z}(\mathfrak{g}_{\rm L}{}'')$ are megaideals of $\mathfrak{g}_{\rm L}$, we have

$$\varphi(\bar{D}^{\rm s}) = c_0\bar{D}^{\rm s} + \bar{P}^x(\chi^0) + \bar{P}^y(\rho^0) + \bar{R}^x(\alpha^0) + \bar{R}^y(\beta^0) + \bar{Z}(\sigma^0) + b_0\bar{P}^v,$$
$$\varphi(\bar{P}^v) = a_1\bar{Z}(1) + b_1\bar{P}^v,$$

where χ^0, ρ^0, α^0, β^0 and σ^0 are smooth functions of t, and c_0, b_0, a_1 and b_1 are constants with $c_0b_1 \neq 0$. Then we evaluate the defining automorphism property at φ and the pair of vector fields $\big(\bar{D}^{\rm s}, \bar{P}^v\big)$,

$$\big[\varphi\big(\bar{D}^{\rm s}\big), \varphi\big(\bar{P}^v\big)\big] = \varphi\big([\bar{D}^{\rm s}, \bar{P}^v]\big) = -\tfrac{3}{2}\varphi\big(\bar{P}^v\big)$$
$$\sim \quad (2c_0 - 1)a_1\bar{Z}(1) + (c_0 - 1)b_1\bar{P}^v = 0,$$

which implies $(2c_0 - 1)a_1 = (c_0 - 1)b_1 = 0$. In view of $b_1 \neq 0$, we derive $c_0 = 1$ and then $a_1 = 0$. In other words, $\varphi(\bar{P}^v) \in \langle \bar{P}^v\rangle =: \bar{\mathfrak{m}}_7$ for any automorphism φ of $\mathfrak{g}_{\rm L}$, i.e., $\bar{\mathfrak{m}}_7$ is a megaideal of $\mathfrak{g}_{\rm L}$. □

The improper megaideal $\mathfrak{g}_{\rm L}$ and the proper megaideals $\mathfrak{g}_{\rm L}{}'$, $\bar{\mathfrak{m}}_2'$ and $\mathfrak{z}(\mathfrak{g}_{\rm L}{}')$ are inessential in the course of constructing the point-symmetry pseudogroup of the system (1.14) using the algebraic method since they are sums of other megaideals, $\mathfrak{g}_{\rm L} = \bar{\mathfrak{m}}_1 + \bar{\mathfrak{m}}_2$, $\mathfrak{g}_{\rm L}{}' = \bar{\mathfrak{m}}_1 + \bar{\mathfrak{m}}_7$, $\bar{\mathfrak{m}}_2' = \bar{\mathfrak{m}}_3 + \bar{\mathfrak{m}}_7$, and $\mathfrak{z}(\mathfrak{g}_{\rm L}{}') = \bar{\mathfrak{m}}_6 + \bar{\mathfrak{m}}_7$. The two constructed one-dimensional megaideals $\bar{\mathfrak{m}}_6$ and $\bar{\mathfrak{m}}_7$ are the most important for further consideration. As for the algebra $\mathfrak{g}$, we cannot of course answer the question of whether the above megaideals exhaust the entire set of proper megaideals of the (infinite-dimensional) algebra $\mathfrak{g}_{\rm L}$.

Theorem 1.13. *The point-symmetry pseudogroup $G_{\rm L}$ of the nonlinear Lax representation* (1.14) *is generated by the transformations of the form*

$$\begin{aligned}&\tilde t = T(t),\quad \tilde x = A^{2/3}T_t^{1/3}x + X^0(t),\quad \tilde y = A^{2/3}T_t^{1/3}y + Y^0(t),\\ &\tilde u = A^2u - \frac{A^2T_{tt}}{18T_t}(x^3+y^3) - \frac{A^{4/3}}{2T_t^{1/3}}(X^0_tx^2+Y^0_ty^2)\\ &\qquad + W^1(t)x + W^2(t)y + W^0(t),\\ &\tilde v = Av + B\end{aligned}$$

and the transformation $\bar{\mathcal{J}}$: $\tilde t = t$, $\tilde x = y$, $\tilde y = x$, $\tilde u = u$, $\tilde v = v$. Here T, X^0, Y^0, W^0, W^1 and W^2 are arbitrary smooth functions of t with $T_t \neq 0$, and A and B are arbitrary constants with $A \neq 0$.

Proof. We follow item (i) of the proof of Theorem 1.3, replacing the point transformation Φ by the point transformation $\bar\Phi$ in the extended space with the coordinates (t,x,y,u,v),

$$\bar\Phi\colon\ (\tilde t,\tilde x,\tilde y,\tilde u,\tilde v) = (T,X,Y,U,V),$$

where (T,X,Y,U,V) is a tuple of smooth functions of variables (t,x,y,u,v) with nonvanishing Jacobian. The condition $\Phi_*\bar P^v \subseteq \bar{\mathfrak{m}}_7$ implies that $T_v = X_v = Y_v = U_v = 0$ and $V_v = \text{const}$. Since $\varpi_*\mathfrak{g}_{\rm L} = \mathfrak{g}$, where ϖ in the natural projection from $\mathbb{R}^5_{t,x,y,u,v}$ onto $\mathbb{R}^4_{t,x,y,u}$, the independence of the tuple (T,X,Y,U) on v means that the t-, x-, y- and u-components of $\bar\Phi$ satisfy all the constraints derived in item (i) of the proof of Theorem 1.3 for the components of the transformation Φ, i.e., they have, up to composing with the transformation $\bar{\mathcal{J}}$: $(\tilde t,\tilde x,\tilde y,\tilde u,\tilde v) = (t,y,x,u,v)$, the form (1.4). It is obvious that the transformation $\mathcal{J}$ is a point symmetry of the system (1.14). Then collecting v-components in the expanded conditions $\Phi_*\bar Z(1) \in \bar{\mathfrak{m}}_6$, $\Phi_*\bar P^z(1) \in \bar{\mathfrak{m}}_3$, $z \in \{x,y\}$, and $\Phi_*\bar D^t(1) \in \bar{\mathfrak{m}}_1$, lead to the equations $V_t =$

$V_x = V_y = V_u = 0$. Hence it has the form $V = Av + B$, where A and B are arbitrary constants with $A \neq 0$.

Each point transformation Φ whose components are of the above form satisfies the condition $\Phi_*\mathfrak{g}_{\rm L} \subseteq \mathfrak{g}_{\rm L}$, which means that this form cannot be constrained more within the framework of the purely algebraic method.

We complete the proof with computing by the direct method. More specifically, using the chain rule, we derive expressions for derivatives of $(\tilde u, \tilde v)$ with respect to $(\tilde t, \tilde x, \tilde y)$ up to order two in terms of the variables and derivatives without tildes, successively substitute the obtained expressions and the expressions for the leading derivatives v_t and v_y in view of the system (1.14) into the system (1.14) written in terms of variables with tildes and split the derived equations with respect to the other (parametric) derivatives of u and v up to order two. The resulting system of equations for parameters of point symmetry transformations of the system (1.14) reduces to the single equation $C^3 = A^2$, i.e., $C = A^{2/3} > 0$. The computation can be simplified by factoring out the transformation $\bar{\mathcal J}$ and the transformations related to varying the pseudogroup parameters T, X^0, Y^0, W^0, W^1, W^2 and B, which are obviously point symmetry transformations of (1.14). In other words, we can set $T = t$, $X^0 = Y^0 = W^0 = W^1 = W^2 = 0$ and $B = 0$. □

Corollary 1.14. *A complete list of discrete point symmetry transformations of the system* (1.14) *that are independent up to composing with each other and with continuous point symmetry transformations of this system is exhausted by three commuting involutions, which can be chosen to be the permutation $\bar{\mathcal J}$ of the variables x and y, $(\tilde t, \tilde x, \tilde y, \tilde u, \tilde v) = (t, y, x, u, v)$, and two transformations $\bar{\mathrm I}^{\rm i}$ and $\bar{\mathrm I}^{v}$ alternating the signs of (t, x, y) and of v, respectively,*

$$\bar{\mathrm I}^{\rm i}\colon (\tilde t, \tilde x, \tilde y, \tilde u, \tilde v) = (-t, -x, -y, u, v) \quad \textit{and}$$
$$\bar{\mathrm I}^{v}\colon (\tilde t, \tilde x, \tilde y, \tilde u, \tilde v) = (t, x, y, u, -v).$$

Therefore, analogously to the pseudogroup G, the quotient group of the point-symmetry pseudogroup $G_{\rm L}$ of the nonlinear Lax representation (1.14) of the dispersionless Nizhnik equation (1.1) with respect to its identity component is isomorphic to the group $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$.

Remark 1.15. The point transformation

$$\bar{\mathcal{J}}^{\rm s}\colon (\tilde t,\tilde x,\tilde y,\tilde u,\tilde v)=(t,-x,-y,-u,v),$$

which is the trivial extension of the discrete point symmetry transformation $\mathcal{J}^{\rm s}$ of the equation (1.1) to v, maps the nonlinear Lax representation (1.14) of the equation (1.1) to an equivalent nonlinear Lax representation of the same equation,

$$v_t=-\frac13\left(v_x^3+\frac{u_{xy}^3}{v_x^3}\right)+u_{xx}v_x+\frac{u_{xy}u_{yy}}{v_x},\quad v_y=\frac{u_{xy}}{v_x}.$$

1.6. Point-symmetry pseudogroup of dispersionless Nizhnik system

The equation (1.1) is in fact a potential equation of the dispersionless counterpart

$$p_t=(h^1p)_x+(h^2p)_y,\quad h^1_y=p_x,\quad h^2_x=p_y \tag{1.15}$$

of the original symmetric Nizhnik system [94, Eq. (4)], cf. Remark 1.1. (We re-denote the dependent variables and scale the system variables for canceling the coefficient 3 on the nonlinear summands and for setting the constant parameters k_1 and k_2 to 1.) Indeed, using the last two equations of the system (1.15) as "short" conservation laws, we introduce the potentials φ^1 and φ^2 defined by the equations

$$\varphi^1_x=h^1,\quad \varphi^1_y=p\quad\text{and}\quad \varphi^2_y=h^2,\quad \varphi^2_x=p.$$

Therefore, we also have the "short" first-level potential conservation law $\varphi^1_y=\varphi^2_x$, for which the associated second-level potential u is defined by the equations with $u_x = \varphi^1$, $u_y = \varphi^2$.[1.4] The dependent variables of the system (1.15) are expressed in terms of the potential u alone, $p = u_{xy}$, $h^1 = u_{xx}$ and $h^2 = u_{yy}$. Substituting these expressions into the first equation of the system (1.15), we derive the equation (1.1) for the potential u.

Since the equation (1.1) and the system (1.15) are related in a nonlocal way, the maximal Lie invariance (pseudo)algebra $\mathfrak{g}_{\rm dN}$ and the point-symmetry pseudogroup $G_{\rm dN}$ the system (1.15) cannot be directly derived from their counterparts $\mathfrak{g}$ and G for the equation (1.1) and should be computed independently. At the same time, each Lie-symmetry vector field Q of (1.1) as belonging to the span of the vector fields (1.2) induces a Lie-symmetry vector field $\hat Q$ of (1.15). The induction map $\mathcal{M}_*\colon \mathfrak{g} \to \mathfrak{g}_{\rm dN}$ is the composition of the standard second prolongation and the projection from the second-order jet space over the basic space $\mathbb{R}^3_{t,x,y} \times \mathbb{R}_u$ onto the space with coordinates (t, x, y, p, h^1, h^2) under the identification $(p, h^1, h^2) = (u_{xy}, u_{xx}, u_{yy})$. It is obvious that $\mathcal{M}_*$ is a Lie-algebra homomorphism with $\ker \mathcal{M}_* = \big\langle R^x(\alpha), R^y(\beta), Z(\sigma)\big\rangle$. The problem is to prove that the homomorphism $\mathcal{M}_*$ is surjective, i.e., it is an epimorphism, $\operatorname{im} \mathcal{M}_* = \mathfrak{g}_{\rm dN}$. This is really the case since computations within the framework of the classical Lie infinitesimal approach show that

$$\mathfrak{g}_{\rm dN} = \big\langle \hat D^t(\tau), \hat D^{\rm s}, \hat P^x(\chi), \hat P^y(\rho)\big\rangle = \mathcal{M}_*\mathfrak{g},$$

where

$$\hat D^t(\tau) = \mathcal{M}_* D^t(\tau), \quad \hat D^{\rm s} = \mathcal{M}_* D^{\rm s},$$
$$\hat P^x(\chi) = \mathcal{M}_* P^x(\chi), \quad \hat P^y(\rho) = \mathcal{M}_* P^y(\rho),$$

that is,

$$\hat D^t(\tau) = \tau\partial_t + \tfrac13\tau_t x\partial_x + \tfrac13\tau_t y\partial_y - \tfrac23\tau_t p\partial_p - \tfrac13(2\tau_t h^1 + \tau_{tt}x)\partial_{h^1}$$
$$-\tfrac13(2\tau_t h^2 + \tau_{tt}y)\partial_{h^2},$$

[1.4] See [116, Section 3.5] and the end of Section VI in [75] for related terminology. The idea of the iterative procedure of introducing potentials can be traced back to [129].

$$\hat{D}^{\rm s} = x\partial_x + y\partial_y + p\partial_p + h^1\partial_{h^1} + h^2\partial_{h^2},$$
$$\hat{P}^x(\chi) = \chi\partial_x - \chi_t\partial_{h^1}, \quad \hat{P}^y(\rho) = \rho\partial_y - \rho_t\partial_{h^2},$$

and the parameter functions τ, χ and ρ run through the set of smooth functions of t.

Since $\mathcal{M}_*$ is a Lie-algebra epimorphism with

$$\ker\mathcal{M}_* = \big\langle R^x(\alpha), R^y(\beta), Z(\sigma)\big\rangle,$$

the nonzero commutation relations between the vector fields spanning $\mathfrak{g}_{\rm dN}$ are exhausted, up to the antisymmetry of the Lie bracket, by

$$\begin{aligned}
&[\hat{D}^t(\tau^1), \hat{D}^t(\tau^2)] = \hat{D}^t(\tau^1\tau^2_t - \tau^1_t\tau^2),\\
&[\hat{D}^t(\tau), \hat{P}^x(\chi)] = \hat{P}^x\big(\tau\chi_t - \tfrac13\tau_t\chi\big),\\
&[\hat{D}^t(\tau), \hat{P}^y(\rho)] = \hat{P}^y\big(\tau\rho_t - \tfrac13\tau_t\rho\big),\\
&[\hat{D}^{\rm s}, \hat{P}^x(\chi)] = -\hat{P}^x(\chi), \quad [\hat{D}^{\rm s}, \hat{P}^y(\rho)] = -\hat{P}^y(\rho).
\end{aligned} \tag{1.16}$$

Accordingly, we were able to construct the following proper megaideals of the algebra $\mathfrak{g}_{\rm dN}$:

$$\begin{aligned}
&\hat{\mathfrak{m}}_1 := \mathfrak{g}_{\rm dN}{}' = \big\langle \hat{D}^t(\tau), \hat{P}^x(\chi), \hat{P}^y(\rho)\big\rangle,\\
&\hat{\mathfrak{m}}_2 := \mathfrak{r}_{\rm dN} = \big\langle \hat{D}^{\rm s}, \hat{P}^x(\chi), \hat{P}^y(\rho)\big\rangle,\\
&\hat{\mathfrak{m}}_3 := \hat{\mathfrak{m}}_2' = \hat{\mathfrak{m}}_1 \cap \hat{\mathfrak{m}}_2 = \big\langle \hat{P}^x(\chi), \hat{P}^y(\rho)\big\rangle.
\end{aligned}$$

By Theorem 1.3, the analogous map $\mathcal{M}\colon G \to G_{\rm dN}$ is a pseudogroup homomorphism, where $\ker\mathcal{M}$ is the pseudosubgroup of G constituted by the transformations of the form (1.4) with $T = t$, $C = 1$ and $X^0 = Y^0 = 0$. Although the problem is again to prove the surjection property of $\mathcal{M}$, the presence of this homomorphism allows us to easily make a conjecture on the general form of point symmetry transformations of the system (1.15), which we then prove using the modified version of the megaideal-based method that was suggested in [85].

Theorem 1.16. *The point-symmetry pseudogroup $G_{\rm dN}$ of the dispersionless Nizhnik system (1.15) is generated by the transformations of the form*

$$
\begin{aligned}
&\tilde t = T(t), \quad \tilde x = CT_t^{1/3}x + X^0(t), \quad \tilde y = CT_t^{1/3}y + Y^0(t),\\
&\tilde p = \frac{C}{T_t^{2/3}}p, \quad \tilde h^1 = \frac{C}{T_t^{2/3}}h^1 - \frac{CT_{tt}}{3T_t^{5/3}}x - \frac{X^0_t}{T_t},\\
&\tilde h^2 = \frac{C}{T_t^{2/3}}h^2 - \frac{CT_{tt}}{3T_t^{5/3}}y - \frac{Y^0_t}{T_t}
\end{aligned}
\tag{1.17}
$$

and the transformation $\hat{\mathfrak J}$: $\tilde t = t$, $\tilde x = y$, $\tilde y = x$, $\tilde p = p$, $\tilde h^1 = h^2$, $\tilde h^2 = h^1$. Here T, X^0 and Y^0 are arbitrary smooth functions of t with $T_t \neq 0$, and C is an arbitrary nonzero constant.

Proof. Although the procedure of proving is in general analogous to that in the proof of Theorem 1.3, computational details are essentially different. Consider a point transformation Φ in the space with the coordinates (t, x, y, p, h^1, h^2),

$$\Phi\colon\ (\tilde t, \tilde x, \tilde y, \tilde p, \tilde h^1, \tilde h^2) = (T, X, Y, P, H^1, H^2),$$

where (T, X, Y, P, H^1, H^2) is a tuple of smooth functions of (t, x, y, p, h^1, h^2) with nonvanishing Jacobian. If it is a point symmetry of the system (1.15), then the pushforward Φ_* of vector fields by Φ satisfies the conditions $\Phi_*\hat{\mathfrak m}_3 \subseteq \hat{\mathfrak m}_3$, $\Phi_*(\hat{\mathfrak m}_1 \setminus \hat{\mathfrak m}_3) \subseteq \hat{\mathfrak m}_1 \setminus \hat{\mathfrak m}_3$ and $\Phi_*(\hat{\mathfrak m}_2 \setminus \hat{\mathfrak m}_3) \subseteq \hat{\mathfrak m}_2 \setminus \hat{\mathfrak m}_3$, and, moreover, $\ker \Phi_* = \{0\}$. For evaluating Φ_*, we choose the following linearly independent vector fields from $\mathfrak g$:

$$
\begin{aligned}
&Q^{1z} := \hat P^z(1), \quad Q^{2z} := \hat P^z(t), \quad Q^{3z} := \hat P^z(t^2),\\
&Q^4 := \hat D^t(1), \quad Q^5 := \hat D^t(t), \quad Q^6 := \hat D^{\rm s}
\end{aligned}
$$

with $z \in \{x, y\}$. Since $Q^{1z}, Q^{2z}, Q^{3z} \in \hat{\mathfrak m}_3$, $Q^4, Q^5 \in \hat{\mathfrak m}_1 \setminus \hat{\mathfrak m}_3$ and $Q^6 \in \hat{\mathfrak m}_2 \setminus \hat{\mathfrak m}_3$, then

$$
\begin{aligned}
&\Phi_*Q^{iz} = \tilde P^x(\tilde\chi^{iz}) + \tilde P^y(\tilde\rho^{iz}), \quad (\tilde\chi^{iz}, \tilde\rho^{iz}) \neq (0,0), \quad i = 1,2,3,\\
&\Phi_*Q^i = \tilde D^t(\tilde\tau^i) + \tilde P^x(\tilde\chi^i) + \tilde P^y(\tilde\rho^i), \quad \tilde\tau^i \neq 0, \quad i = 4,5,\\
&\Phi_*Q^i = \lambda^i\tilde D^{\rm s} + \tilde P^x(\tilde\chi^i) + \tilde P^y(\tilde\rho^i), \quad \lambda^i \neq 0, \quad i = 6.
\end{aligned}
\tag{1.18}
$$

Let $(1.18)_{iz}$, $i=1,2,3$, and $(1.18)_i$, $i=4,5,6$, refer to the ith equation in the system (1.18) with $z\in\{x,y\}$ for $i=1,2,3$.

The identity $\Phi_*Q^{3z}-2\Phi_*(t)\Phi_*Q^{2z}+\Phi_*(t^2)\Phi_*Q^{1z}=0$ and the corresponding combination of the equations $(1.18)_{1z}$, $(1.18)_{2z}$ and $(1.18)_{3z}$ imply the system

$$\tilde\chi^{3z}(T)-2t\tilde\chi^{2z}(T)+t^2\tilde\chi^{1z}(T)=0,\quad \tilde\chi^{3z}_{\tilde t}(T)-2t\tilde\chi^{2z}_{\tilde t}(T)+t^2\tilde\chi^{1z}_{\tilde t}(T)=0,$$
$$\tilde\rho^{3z}(T)-2t\tilde\rho^{2z}(T)+t^2\tilde\rho^{1z}(T)=0,\quad \tilde\rho^{3z}_{\tilde t}(T)-2t\tilde\rho^{2z}_{\tilde t}(T)+t^2\tilde\rho^{1z}_{\tilde t}(T)=0,$$

whose differential consequences are $\tilde\chi^{2z}(T)=t\tilde\chi^{1z}(T)$ and $\tilde\rho^{2z}(T)=t\tilde\rho^{1z}(T)$. Similarly to the previous proofs, we derive from these equations that $T=T(t)$ with $T_t\neq0$. We collect the components in the equations $(1.18)_{1z}$ and in the combination $\Phi_*(t)(1.18)_{1z}-(1.18)_{2z}$, deriving the constraints

$$X_z=\tilde\chi^{1z}(T),\quad Y_z=\tilde\rho^{1z}(T),\quad P_z=0,\quad X_{h^z}=Y_{h^z}=P_{h^z}=0,$$
$$H^1_z=-\tilde\chi^{1z}_{\tilde t}(T)=-\frac{X_{zt}}{T_t},\quad H^2_z=-\tilde\rho^{1z}_{\tilde t}(T)=-\frac{Y_{zt}}{T_t},$$
$$H^1_{h^z}=\tilde\chi^{2z}_{\tilde t}(T)-t\tilde\chi^{1z}_{\tilde t}(T)=\frac{\tilde\chi^{1z}(T)}{T_t}=\frac{X_z}{T_t},$$
$$H^2_{h^z}=\tilde\rho^{2z}_{\tilde t}(T)-t\tilde\rho^{1z}_{\tilde t}(T)=\frac{\tilde\rho^{1z}(T)}{T_t}=\frac{Y_z}{T_t}$$

with $h^x:=h^1$ and $h^y:=h^2$. Hence

$$\begin{aligned}
&X=X^1(t)x+X^2(t)y+X^0(t,p),\\
&Y=Y^1(t)x+Y^2(t)y+Y^0(t,p),\quad P=P(t,p),\\
&H^1=\frac{X^1}{T_t}h^1+\frac{X^2}{T_t}h^2-\frac{X^1_t}{T_t}x-\frac{X^2_t}{T_t}y+\breve H^1(t,p),\\
&H^2=\frac{Y^1}{T_t}h^1+\frac{Y^2}{T_t}h^2-\frac{Y^1_t}{T_t}x-\frac{Y^2_t}{T_t}y+\breve H^2(t,p),
\end{aligned}\tag{1.19}$$

where X^1, X^2, X^0, Y^1, Y^2, Y^0, P, $\breve H^1$ and $\breve H^2$ are sufficiently smooth functions of their arguments with $P_p(X^1Y^2-X^2Y^1)\neq0$.

The componentwise splitting of $(1.18)_4$, $3(1.18)_5-3\Phi_*(t)(1.18)_4$ and $(1.18)_6$ leads to the system

$$T_t=\tilde\tau^4(T),\quad X_t=\frac13\tilde\tau^4_{\tilde t}(T)X+\tilde\chi^4(T),$$

$$
\begin{aligned}
&Y_t = \frac{1}{3}\tilde{\tau}^4_{\tilde{t}}(T)Y + \tilde{\rho}^4(T), \quad P_t = -\frac{2}{3}\tilde{\tau}^4_{\tilde{t}}(T)P,\\
&H^1_t = -\frac{2}{3}\tilde{\tau}^4_{\tilde{t}}(T)H^1 - \frac{1}{3}\tilde{\tau}^4_{\tilde{t}\tilde{t}}(T)X - \tilde{\chi}^4_{\tilde{t}}(T),\\
&H^2_t = -\frac{2}{3}\tilde{\tau}^4_{\tilde{t}}(T)H^2 - \frac{1}{3}\tilde{\tau}^4_{\tilde{t}\tilde{t}}(T)Y - \tilde{\rho}^4_{\tilde{t}}(T),
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\tau}^5(T) = t\tilde{\tau}^4(T), \quad xX_x + yX_y - 2pX_p = X + 3\tilde{\chi}^5(T) - 3t\tilde{\chi}^4(T),\\
&xY_x + yY_y - 2pY_p = Y + 3\tilde{\rho}^5(T) - 3t\tilde{\rho}^4(T), \quad pP_p = P,\\
&xH^1_x + yH^1_y - 2pH^1_p - 2h^1H^1_{h^1} - 2h^2H^1_{h^2} =\\
&\quad -2H^1 + (T_t^{-1})_tX - 3\tilde{\chi}^5_{\tilde{t}}(T) + 3t\tilde{\chi}^4_{\tilde{t}}(T),\\
&xH^2_x + yH^2_y - 2pH^2_p - 2h^1H^2_{h^1} - 2h^2H^2_{h^2} =\\
&\quad -2H^2 + (T_t^{-1})_tY - 3\tilde{\rho}^5_{\tilde{t}}(T) + 3t\tilde{\rho}^4_{\tilde{t}}(T),
\end{aligned}
$$

$$
\begin{aligned}
&xX_x + yX_y + pX_p = \lambda^6X + \tilde{\chi}^6(T),\\
&xY_x + yY_y + pY_p = \lambda^6Y + \tilde{\rho}^6(T), \quad pP_p = \lambda^6P,\\
&xH^1_x + yH^1_y + pH^1_p + h^1H^1_{h^1} + h^2H^1_{h^2} = \lambda^6H^1 - \tilde{\chi}^6_{\tilde{t}}(T),\\
&xH^2_x + yH^2_y + pH^2_p + h^1H^2_{h^1} + h^2H^2_{h^2} = \lambda^6H^2 - \tilde{\rho}^6_{\tilde{t}}(T).
\end{aligned}
$$

Here we at once take into account the constraints $\tilde{\tau}^4(T) = T_t$ and $\tilde{\tau}^5(T) = tT_t$, in view of which we have

$$
\begin{aligned}
&\tilde{\tau}^4_{\tilde{t}}(T) = T_{tt}/T_t, \quad \tilde{\tau}^5_{\tilde{t}}(T) - t\tilde{\tau}^4_{\tilde{t}}(T) = 1,\\
&\tilde{\tau}^4_{\tilde{t}\tilde{t}}(T) = (T_{tt}/T_t)_t/T_t, \quad \tilde{\tau}^5_{\tilde{t}\tilde{t}}(T) - t\tilde{\tau}^4_{\tilde{t}\tilde{t}}(T) = -(T_t^{-1})_t.
\end{aligned}
$$

We substitute the earlier derived form (1.19) of the components of Φ into the above system, split the expanded system with respect to (x, y, h^1, h^2) and solve the obtained system of constraints

$$
\begin{aligned}
&X^j_t = \frac{T_{tt}}{3T_t}X^j, \quad Y^j_t = \frac{T_{tt}}{3T_t}Y^j, \quad j = 1, 2,\\
&X^0_t = \frac{T_{tt}}{3T_t}X^0 + \tilde{\chi}^4(T), \quad Y^0_t = \frac{T_{tt}}{3T_t}Y^0 + \tilde{\rho}^4(T),\\
&P_t = -\frac{2T_{tt}}{3T_t}P, \quad \breve{H}^1_t = -\frac{2T_{tt}}{3T_t}\breve{H}^1 - \left(\frac{T_{tt}}{T_t}\right)_t\frac{X^0}{3T_t} - \tilde{\chi}^4_{\tilde{t}}(T),
\end{aligned}
$$

$$
\begin{aligned}
&\breve H^2_t=-\frac{2T_{tt}}{3T_t}\breve H^2-\left(\frac{T_{tt}}{T_t}\right)_t\frac{Y^0}{3T_t}-\tilde\rho^4_{\tilde t}(T),\\
&pX^0_p=X^0+\tilde\chi^6(T),\quad -2pX^0_p=X^0+3\tilde\chi^5(T)-3t\tilde\chi^4(T),\\
&pY^0_p=Y^0+\tilde\rho^6(T),\quad -2pY^0_p=Y^0+3\tilde\rho^5(T)-3t\tilde\rho^4(T),\\
&pP_p=P,\quad \lambda^6=1,\\
&p\breve H^1_p=\breve H^1-\tilde\chi^6_{\tilde t}(T),\\
&3xH^1_x+3yH^1_y=(T_t^{-1})_tX-2\tilde\chi^6_{\tilde t}(T)-3\tilde\chi^5_{\tilde t}(T)+3t\tilde\chi^4_{\tilde t}(T),\\
&p\breve H^2_p=\breve H^2-\tilde\rho^6_{\tilde t}(T),\\
&3xH^2_x+3yH^2_y=(T_t^{-1})_tY-2\tilde\rho^6_{\tilde t}(T)-3\tilde\rho^5_{\tilde t}(T)+3t\tilde\rho^4_{\tilde t}(T).
\end{aligned}
$$

This system implies that in fact $X^0=X^0(t)$ and $Y^0=Y^0(t)$. The other its independent consequences are only

$$
\begin{aligned}
&\tilde\chi^4(T)=X^0_t-\frac{T_{tt}}{3T_t}X^0,\quad \tilde\rho^4(T)=Y^0_t-\frac{T_{tt}}{3T_t}Y^0,\\
&3\big(\tilde\chi^5(T)-t\tilde\chi^4(T)\big)=\tilde\chi^6(T)=-X^0,\\
&3\big(\tilde\rho^5(T)-t\tilde\rho^4(T)\big)=\tilde\rho^6(T)=-Y^0,\\
&X^j_t=\frac{T_{tt}}{3T_t}X^j,\ \ Y^j_t=\frac{T_{tt}}{3T_t}Y^j,\ \ j=1,2,\quad P_t=-\frac{2T_{tt}}{3T_t}P,\quad pP_p=P,\\
&p\breve H^1_p=\breve H^1+\frac{X^0_t}{T_t},\quad \breve H^1_t=-\frac{2T_{tt}}{3T_t}\breve H^1-\frac{X^0_{tt}}{T_t}+\frac{T_{tt}}{3T_t^2}X^0_t,\\
&p\breve H^2_p=\breve H^2+\frac{Y^0_t}{T_t},\quad \breve H^2_t=-\frac{2T_{tt}}{3T_t}\breve H^2-\frac{Y^0_{tt}}{T_t}+\frac{T_{tt}}{3T_t^2}Y^0_t.
\end{aligned}
$$

The equations for the parameter functions involved in Φ integrate to

$$
\begin{aligned}
&X^j=A_jT_t^{1/3},\ \ Y^j=B_jT_t^{1/3},\ \ P=\frac{Cp}{T_t^{2/3}},\\
&\breve H^1=\frac{E_1p}{T_t^{2/3}}-\frac{X^0_t}{T_t},\ \ \breve H^2=\frac{E_2p}{T_t^{2/3}}-\frac{Y^0_t}{T_t},
\end{aligned}
\tag{1.20}
$$

where A_j, B_j, C and E_j, $j=1,2$, are constants with $C(A_1B_2-A_2B_1)\neq0$.

The transformations defined by (1.19)–(1.20) constitute a pseudogroup $\mathfrak G$, which contains the set $\mathfrak N$ of the transformations of the form (1.17) with $C = 1$ as a normal pseudosubgroup. The later transformations are point symmetries of the system (1.15), which can be easily checked by the direct method although it is also clear due to the observation that they are generated by Lie symmetries of the system (1.15) and the time reflection $(\tilde t, \tilde x, \tilde y, \tilde p, \tilde h^1, \tilde h^2) = (-t, -x, -y, p, h^1, h^2)$. The pseudogroup $\mathfrak G$ splits over $\mathfrak N$, $\mathfrak G = \mathfrak H \ltimes \mathfrak N$, where the subgroup $\mathfrak H$ of $\mathfrak G$ consists of the transformations of the form

$$
\begin{aligned}
&T = t, \quad X = A_1x + A_2y, \quad Y = B_1x + B_2y,\\
&P = Cp, \quad H^1 = A_1h^1 + A_2h^2 + E_1p, \quad H^2 = B_1h^1 + B_2h^2 + E_2p,
\end{aligned}
$$

A_j, B_j, C and E_j, $j = 1, 2$, are arbitrary constants with $C(A_1B_2 - A_2B_1) \neq 0$. Therefore, we can factor out the transformations from $\mathfrak N$ and consider only the transformations from $\mathfrak H$ in the remainder of the proof.

It is easy to check that $\Psi_*\mathfrak g_{\rm dN} = \mathfrak g_{\rm dN}$ for any $\Psi \in \mathfrak H$. This means that no constraints for the above constant parameters can be found within the algebraic approach. Therefore, for completing the proof, the direct method should necessarily be applied. The computation is standard. The chain rule implies expressions for first-order derivatives of $(\tilde p, \tilde h^1, \tilde h^2)$ with respect to $(\tilde t, \tilde x, \tilde y)$ in terms of the variables and derivatives without tildes, which we substitute jointly with the expressions for $(\tilde p, \tilde h^1, \tilde h^2)$ and, e.g., the expressions for the derivatives p_t, h^1_y and h^2_x according to the system (1.15) into the system (1.15) written in terms of variables with tildes and split the derived equations with respect to the other (parametric) first-order derivatives of (p, h^1, h^2). As a result, we derive the system

$$
\begin{aligned}
&A_1A_2 = B_1B_2 = 0, \quad B_1E_1 = A_1E_2, \quad B_2E_1 = A_2E_2,\\
&A_1^2 - E_1A_2 = CB_2, \quad A_2^2 - E_1A_1 = CB_1,\\
&B_1^2 - E_2B_2 = CA_2, \quad B_2^2 - E_2B_1 = CA_1.
\end{aligned}
$$

In view of the inequality $C(A_1B_2 - A_2B_1) \neq 0$, it implies that $E_1 = E_2 = 0$ and $(A_1, A_2, B_1, B_2) \in \big\{(C, 0, 0, C),\ (0, C, C, 0)\big\}$. □

Corollary 1.17. $\mathcal{M}_*\mathfrak{g} = \mathfrak{g}_{\rm dN}$ *and* $\mathcal{M}G = G_{\rm dN}$*. In other words, the maximal Lie invariance algebra* $\mathfrak{g}_{\rm dN}$ *and the point-symmetry pseudogroup* $G_{\rm dN}$ *the system* (1.15) *are induced by their counterparts* $\mathfrak{g}$ *and* G *for the equation* (1.1)*, respectively.*

Corollary 1.18. *A complete list of discrete point symmetry transformations of the system* (1.15) *that are independent up to composing with each other and with continuous point symmetry transformations of this equation is exhausted by three commuting involutions, which can be chosen to be the permutation* $\hat{\mathfrak{J}}$ *of the variables* x *and* y*, and two transformations* $\hat{\mathfrak{I}}^{\rm i}$ *and* $\hat{\mathfrak{I}}^{\rm s}$ *alternating the signs of* (t,x,y) *and of* (x,y,p,h^1,h^2)*, respectively,*

$$\hat{\mathfrak{J}}\colon (\tilde t,\tilde x,\tilde y,\tilde p,\tilde h^1,\tilde h^2) = (t,y,x,p,h^2,h^1),$$

$$\hat{\mathfrak{I}}^{\rm i}\colon (\tilde t,\tilde x,\tilde y,\tilde p,\tilde h^1,\tilde h^2) = (-t,-x,-y,p,h^1,h^2),$$

$$\hat{\mathfrak{I}}^{\rm s}\colon (\tilde t,\tilde x,\tilde y,\tilde p,\tilde h^1,\tilde h^2) = (t,-x,-y,-p,-h^1,-h^2).$$

Hence again the quotient group of the point-symmetry pseudogroup $G_{\rm dN}$ of the dispersionless Nizhnik system (1.15) with respect to the identity component of this pseudogroup is isomorphic to the group $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$.

1.7. Defining geometric properties

We find geometric properties of the dispersionless Nizhnik equation (1.1) that completely define this equation. In this section, by u_κ or by $u_{\kappa_0\kappa_1\kappa_2}$ with the multi-index $\kappa=(\kappa_0,\kappa_1,\kappa_2)\in\mathbb{N}_0^3$ we denote the jet variable that is associated with the derivative $\partial^{\kappa_0+\kappa_1+\kappa_2}u/\partial t^{\kappa_0}\partial x^{\kappa_1}\partial y^{\kappa_2}$.

Lemma 1.19. *A partial differential equation of order less than or equal to three with three independent variables is invariant with respect to the algebra* $\mathfrak{g}$ *if and only if it is of the form*

$$u_{txy} = (u_{xx}u_{xy})_x + (u_{xy}u_{yy})_y + u_{xy}u_{xyy}H\left(\frac{u_{xxx}-u_{yyy}}{u_{xyy}},\frac{u_{xxy}}{u_{xyy}}\right), \quad (1.21)$$

where H *is an arbitrary smooth function of its arguments.*

Proof. The (infinite) prolongations $Q_{(\infty)}$ of the vector fields Q, which are presented in (1.2) and span the maximal Lie invariance (pseudo)algebra $\mathfrak{g}$ of the equation (1.1), are

$$
\begin{aligned}
D^t_{(\infty)}(\tau) &= \tau\partial_t + \frac{1}{3}\tau_t x\partial_x + \frac{1}{3}\tau_t y\partial_y \\
&\quad - \sum_{k=2}^{\infty}\tau^{(k)}\bigg(\frac{x^3+y^3}{18}\partial_{u_{k-2,00}} + \frac{x^2}{6}\partial_{u_{k-2,10}} + \frac{x}{3}\partial_{u_{k-2,20}} \\
&\qquad\qquad + \frac{1}{3}\partial_{u_{k-2,30}} + \frac{y^2}{6}\partial_{u_{k-2,01}} + \frac{y}{3}\partial_{u_{k-2,02}} + \frac{1}{3}\partial_{u_{k-2,03}}\bigg) \\
&\quad - \sum_{\kappa}\sum_{k=1}^{\infty}\tau^{(k)}\left(\binom{\kappa_0}{k} + \frac{\kappa_1+\kappa_2}{3}\binom{\kappa_0}{k-1}\right)u_{\kappa_0+1-k,\kappa_1\kappa_2}\partial_{u_\kappa} \\
&\quad - \frac{1}{3}\sum_{\kappa}\sum_{k=2}^{\infty}\binom{\kappa_0}{k-1}\tau^{(k)}\big(xu_{\kappa_0+1-k,\kappa_1+1,\kappa_2} \\
&\qquad\qquad + yu_{\kappa_0+1-k,\kappa_1,\kappa_2+1}\big)\partial_{u_\kappa}, \\
D^{\mathrm{s}}_{(\infty)} &= x\partial_x + y\partial_y + \sum_{\kappa}(3-\kappa_1-\kappa_2)u_\kappa\partial_{u_\kappa}, \\
P^x_{(\infty)}(\chi) &= \chi\partial_x - \sum_{k=1}^{\infty}\chi^{(k)}\bigg(\frac{x^2}{2}\partial_{u_{k-1,00}} + x\partial_{u_{k-1,10}} + \partial_{u_{k-1,20}} \\
&\qquad\qquad + \sum_{\kappa}\binom{\kappa_0}{k}u_{\kappa_0-k,\kappa_1+1,\kappa_2}\partial_{u_\kappa}\bigg), \\
P^y_{(\infty)}(\rho) &= \rho\partial_y - \sum_{k=1}^{\infty}\rho^{(k)}\bigg(\frac{y^2}{2}\partial_{u_{k-1,00}} + y\partial_{u_{k-1,01}} + \partial_{u_{k-1,02}} \\
&\qquad\qquad + \sum_{\kappa}\binom{\kappa_0}{k}u_{\kappa_0-k,\kappa_1,\kappa_2+1}\partial_{u_\kappa}\bigg), \\
R^x_{(\infty)}(\alpha) &= \sum_{k=0}^{\infty}\alpha^{(k)}\big(x\partial_{u_{k00}} + \partial_{u_{k10}}\big), \\
R^y_{(\infty)}(\beta) &= \sum_{k=0}^{\infty}\beta^{(k)}\big(y\partial_{u_{k00}} + \partial_{u_{k01}}\big), \\
Z_{(\infty)}(\sigma) &= \sum_{k=0}^{\infty}\sigma^{(k)}\partial_{u_{k00}}.
\end{aligned}
$$

Recall that run through the set of smooth functions of t, $\binom{n}{k} := 0$ if $k > n$. The differential invariants of the identity component $G_{\rm id}$ of the point-symmetry pseudogroup G of the equation (1.1) can be found as differential functions F of u that satisfy the equations $Q_{(\infty)}F = 0$, where Q runs through the set of the vector fields (1.2). We can split these equations with respect to the derivatives of the parameter functions τ, χ, ρ, α, β and σ and then, after deriving the equations $F_x = F_y = 0$, with respect to x and y, which leads to the following system for F:

$$
\begin{aligned}
&F_t = F_x = F_y = F_{u_{k00}} = F_{u_{k10}} = F_{u_{k01}} = 0,\\
&F_{u_{k20}} + \sum_\kappa \binom{\kappa_0}{k+1} u_{\kappa_0-k-1,\kappa_1+1,\kappa_2} F_{u_\kappa} = 0,\\
&F_{u_{k02}} + \sum_\kappa \binom{\kappa_0}{k+1} u_{\kappa_0-k-1,\kappa_1,\kappa_2+1} F_{u_\kappa} = 0,\\
&\sum_\kappa (3-\kappa_1-\kappa_2) u_\kappa F_{u_\kappa} = 0, \quad \sum_\kappa (\kappa_0+1) u_\kappa F_{u_\kappa} = 0,\\
&F_{u_{k30}} + F_{u_{k03}} + \sum_\kappa \left(3\binom{\kappa_0}{k+2} + (\kappa_1+\kappa_2)\binom{\kappa_0}{k+1}\right)\\
&\qquad \times u_{\kappa_0-k-1,\kappa_1\kappa_2} F_{u_\kappa} = 0, \ \ k \in \mathbb{N}_0.
\end{aligned}
$$

Let the order of F as a differential function be less than or equal to three. Then the equations in the first row mean that F is a function at most u_{xx}, u_{xy}, u_{yy}, u_{txx}, u_{txy}, u_{tyy}, u_{xxx}, u_{xxy}, u_{xyy} and u_{yyy}. Then the equations in the second row with $k = 1$ and with $k = 0$ successively imply

$$
F_{u_{txx}} = F_{u_{tyy}} = 0, \quad F_{u_{xx}} + u_{xxy} F_{u_{txy}} = 0, \quad F_{u_{yy}} + u_{xyy} F_{u_{txy}} = 0.
$$

The equations in the third row and the equation in the last row with $k = 0$ reduce to

$$
\begin{aligned}
&u_{xx} F_{u_{xx}} + u_{xy} F_{u_{xy}} + u_{yy} F_{u_{yy}} + u_{txy} F_{u_{txy}} = 0,\\
&u_{xxx} F_{u_{xxx}} + u_{xxy} F_{u_{xxy}} + u_{xyy} F_{u_{xyy}} + u_{yyy} F_{u_{yyy}} + u_{txy} F_{u_{txy}} = 0,\\
&F_{u_{xxx}} + F_{u_{yyy}} + 2u_{xy} F_{u_{txy}} = 0.
\end{aligned}
$$

The other equations are satisfied identically in view of the derived equations. Integrating the latter equations, we obtain that any differential invariant F of order less than or equal to three of the group $G_{\rm id}$ is a function of

$$\omega_0 := \frac{u_{txy} - (u_{xx}u_{xy})_x - (u_{xy}u_{yy})_y}{u_{xy}u_{xyy}}, \quad \omega_1 := \frac{u_{xxx} - u_{yyy}}{u_{xyy}}, \quad \omega_2 := \frac{u_{xxy}}{u_{xyy}},$$

$F = F(\omega_0, \omega_1, \omega_2)$. Hence the group $G_{\rm id}$ admits no differential invariants of orders zero, one and two. The above consideration also implies that the group $G_{\rm id}$ admits no codimension-one singular invariant manifolds in the third-order jet space $\mathrm{J}^3(\mathbb{R}^3_{txy} \times \mathbb{R}_u)$. Therefore, a partial differential equation for the unknown function u depending on (t, x, y) is $G_{\rm id}$-invariant if and only if it is of the form $F(\omega_0, \omega_1, \omega_2) = 0$, where $F_{\omega_0} \neq 0$ since otherwise the variable t is not significant and rather plays the role of a parameter, and (1.21) is an equivalent form for such equations. □

Any equation of the form (1.21) is invariant with respect to the point transformations $\mathcal{I}^{\rm i}$ and $\mathcal{I}^{\rm s}$ alternating the signs of (t, x, y) and of (x, y, u), respectively, cf. Corollary 1.4. At the same time, the permutation $\mathcal{J}$ of the variables x and y is a point symmetry transformation of such an equation if and only if

$$H(\omega_1, \omega_2) = \omega_2 H(-\omega_2^{-1}\omega_1, \omega_2^{-1}).$$

The space of local conservation laws of the dispersionless Nizhnik equation (1.1) is infinite-dimensional, and the simplest conservation-law characteristics of this equation are 1, u_{xx} and u_{yy}. Let us check when an equation of the form (1.21) admits these conservation-law characteristics.

Lemma 1.20. *(i) An equation of the form* (1.21) *admits the conservation-law characteristic* 1 *and thus it is in conserved form if and only if H is an affine function of (ω_1, ω_2), i.e., $H = a\omega_1 + b\omega_2 + c$ for some constants a, b and c, and the equation takes the form*

$$\begin{aligned} u_{txy} = {} & (u_{xx}u_{xy})_x + (u_{xy}u_{yy})_y \\ & + u_{xy}\big(a(u_{xxx} - u_{yyy}) + bu_{xxy} + cu_{xyy}\big). \end{aligned} \tag{1.22}$$

(ii) An equation of the form (1.22) *admits the conservation-law characteristic* u_{xx} *or* u_{yy} *if and only if* $a = b = 0$ *or* $a = c = 0$*, respectively.*

Proof. The differential function

$$N := u_{txy} - (u_{xx}u_{xy})_x - (u_{xy}u_{yy})_y - u_{xy}u_{xyy}H(\omega_1, \omega_2)$$

is (locally) a total divergence if and only if $\mathsf{E}N = 0$, where E is the Euler operator with respect to u,

$$\mathsf{E} := \sum_{\kappa} (-\mathrm{D}_t)^{\kappa_0}(-\mathrm{D}_x)^{\kappa_1}(-\mathrm{D}_y)^{\kappa_2}\partial_{u_\kappa},$$

see, e.g., [96, Theorem 4.7]. Collecting coefficients of sixth-order derivatives of u in the equation $\mathsf{E}N = 0$, we derive the system

$$H_{\omega_1\omega_1} = H_{\omega_1\omega_2} = H_{\omega_2\omega_2} = 0,$$

and, in view of this system, the equation $\mathsf{E}N = 0$ is satisfied identically. This proves (i).

To prove (ii), it suffices to similarly consider the equations $\mathsf{E}(u_{xx}N) = 0$ and $\mathsf{E}(u_{yy}N) = 0$ for affine functions H of (ω_1, ω_2). □

Lemmas 1.19 and 1.20 jointly imply the following theorem.

Theorem 1.21. *An* r*th order* ($r \in \{1, 2, 3\}$) *partial differential equation with three independent variables admits the algebra* $\mathfrak{g}$ *as its Lie invariance algebra and the conservation-law characteristics* 1, u_{xx} *and* u_{yy} *if and only if it coincides with the dispersionless Nizhnik equation* (1.1)*.*

In view of Theorem 1.21, the invariance with respect to the algebra $\mathfrak{g}$ and admitting the conservation-law characteristics 1, u_{xx} and u_{yy} lead to the invariance with respect to the entire group G, which includes the discrete point symmetry transformations $\mathcal{J}$, $\mathcal{J}^{\mathrm{i}}$ and $\mathcal{J}^{\mathrm{s}}$, and to admitting the entire (infinite-dimensional) space of conservation-law characteristics of the equation (1.1).

1.8. Conclusion

Let us discuss some implications of the results of this chapter in the form of a chain of remarks.

Remark 1.22. Item (ii) of the proof of Theorem 1.3 is the first example of applying the megaideal-based version of the algebraic method to computing the contact-symmetry (pseudo)group of a partial differential equation in the literature. An example of computing contact symmetries of a partial differential equation using the automorphism-based version of the algebraic method was presented in [61].

Remark 1.23. Item (i) of the proof of Theorem 1.3 shows that the conditions (1.5) exhaustively define the point-symmetry pseudogroup G of the equation (1.1), which is the first example of such a kind in the literature. In other words, the second part of the computation procedure of the algebraic method using the direct method is a trivial check that all the singled out point transformations, which are either of the form (1.4) or compositions of transformations of the form (1.4) with the transformation $\mathscr{J}$, are indeed symmetries of the equation (1.1). In view of item (ii) of the proof of Theorem 1.3, the same claim is relevant for the contact-symmetry pseudogroup $G_{\rm c}$ of (1.1) as well. At the same time, this is not the case for the point-symmetry pseudogroup $G_{\rm L}$ of the nonlinear Lax representation (1.14) of the equation (1.1) and even more so for the point-symmetry pseudogroup $G_{\rm dN}$ of the system (1.15), which is nonlocally related to the equation (1.1).

Remark 1.24. It is obvious that $\mathscr{J}$, $\mathscr{I}^{\rm i}$ and $\mathscr{I}^{\rm s}$ are point symmetries of the equation (1.1), and the identity component of the pseudogroup G, whose infinitesimal counterpart is the algebra $\mathfrak{g}$, consists of the point transformations of the form (1.4) with $T_t > 0$ and $C > 0$. Therefore, all the transformations described in Theorem 1.3 are point symmetries of (1.1), and the first prolongation of these transformations are contact symmetries of (1.1).

At the same time, this is a simple part of the statement of Theorem 1.3 although it is still not too trivial as shown by the imprecise formulation of its analog in [92]. In fact, the purpose of the proof of Theorem 1.3 is to check that the equation (1.1) admits no other point and contact symmetry transformations.

Remark 1.25. As noted at the end of Section 1.1, the nonzero improper megaideal of $\mathfrak{g}$, which is the entire algebra $\mathfrak{g}$ itself, can be neglected in the course of applying the megaideal-based method to computing the point-symmetry pseudogroup G of the equation (1.1) since it is the sum of two proper megaideals, $\mathfrak{g} = \mathfrak{m}_1 + \mathfrak{m}_2$. This is not the case for the megaideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Nevertheless, if we use one of them, then the condition $\Phi_*\mathfrak{m} \subseteq \mathfrak{m}$ for the other implies no new constraints for the transformation components, and the megaideal set $\{\mathfrak{m}_2, \dots, \mathfrak{m}_6\}$ assures a bit more effective and simpler computations than $\{\mathfrak{m}_1, \mathfrak{m}_3, \dots, \mathfrak{m}_6\}$. It is not yet clear how to a priori identify megaideals whose involvement in the computation is not too essential although they are not sums of other proper megaideals.

Remark 1.26. The span of each of the sets of linearly independent vector fields that were selected for use in the course of applying the megaideal-based version of the algebraic method in question in the present thesis and in [85] is closed with respect to the Lie bracket, i.e., it is a subalgebra of the corresponding invariance algebra. It is still not known whether this property plays a certain role and whether its appearance is an occasional phenomenon or appropriate vector fields can be always chosen in the way to possess it.

Remark 1.27. The selected sets of linearly independent vector fields are unexpectedly small but still allow us to effectively compute the corresponding point- and contact-symmetry groups, especially when involving megaideals. Nevertheless, we do not know whether the cardinalities of these sets are minimum. In general, one has developed no techniques that would help to a priori estimate sufficient numbers of such vector fields, not to mention finding the minimum among these numbers and the optimal selection of vector fields for simplifying computations.

Remark 1.28. In the course of computing the point-symmetry pseudogroup G of the equation (1.1), it is optimal and sufficient to use the conditions $\Phi_*(\mathfrak{s}_1\cap\mathfrak{m}_j)\subseteq\mathfrak{m}_j$, $j=2,\dots,6$, which jointly implies the condition $\Phi_*\mathfrak{g}\subseteq\mathfrak{g}$. We have additionally checked that for each $k\in\{2,\dots,6\}$, the collection of the conditions $\Phi_*(\mathfrak{s}_1\cap\mathfrak{m}_j)\subseteq\mathfrak{m}_j$, $j\in M_k$, where $M_6=\{6\}$, $M_5=\{5\}$, $M_4=\{4,5\}$, $M_3=\{3,4,5\}$, $M_2=\{2,3,4,5\}$ and $M_1=\{2,3,4,5,6\}$, implies the condition $\Phi_*\mathfrak{m}_k\subseteq\mathfrak{m}_k$. As noted in Remark 1.25, using the subalgebra $\mathfrak{s}_2$ leads to a bit more complicated computations but allows us to replace the megaideal $\mathfrak{m}_2$ with $\mathfrak{m}_1$. Moreover, it suffices to consider the conditions $\Phi_*(\mathfrak{s}_2\cap\mathfrak{m}_j)\subseteq\mathfrak{m}_j$, $j=1,3,4,5$, which jointly implies the conditions $\Phi_*\mathfrak{g}\subseteq\mathfrak{g}$, and thus $\Phi_*\mathfrak{m}_k\subseteq\mathfrak{m}_k$, $k\in\{1,2,6\}$, as well.

Remark 1.29. In the course of proving Theorem 1.7, we have checked which subalgebras of $\mathfrak{g}$ among $\mathfrak{s}_1$, $\bar{\mathfrak{s}}_1$ and $\mathfrak{s}_2$ define diffeomorphisms stabilizing $\mathfrak{g}$. Simultaneously, we have in fact recomputed the group G only using the condition $\Phi_*\mathfrak{s}_2\subseteq\mathfrak{g}$, i.e., involving no proper megaideals of $\mathfrak{g}$. Although the recomputation is based on a technique analogous to that in the proof of Theorem 1.3, it is much more complicated. Moreover, in contrast to the condition with $\mathfrak{s}_2$, the analogous condition with $\bar{\mathfrak{s}}_1$ or, moreover, with $\mathfrak{s}_1$ does not imply the complete system of determining equations for point symmetries of (1.1). The situation is even more dramatic in the case of contact transformations. From the condition $\Psi_*(\mathfrak{s}_4\cap\mathfrak{m}_{j(1)})\subseteq\mathfrak{m}_{j(1)}$, $j=4,5$, for a contact transformation Ψ, where $\mathfrak{s}_4=\langle Z(1),Z(t),R^x(1),R^y(1)\rangle$ is the common four-dimensional subalgebra of $\mathfrak{s}_1$ and $\mathfrak{s}_2$, it is easy to derive that this transformation is the first prolongation of a point transformation in the space with the coordinates (t,x,y,u). This is the content of item (ii) of the proof of Theorem 1.3. We do not know whether this property of Ψ follows even from the condition $\Psi_*(\mathfrak{s}_1\cup\mathfrak{s}_2)_{(1)}\subseteq\mathfrak{g}_{(1)}$. This weakened condition, which does not involve the knowledge of megaideals of $\mathfrak{g}$, implies a too complicated system of equations for the components of Ψ, and techniques applied in the present thesis are not appropriate for solving such a system. This is the reason why in Theorem 1.8 we have used the subalgebra $\mathfrak{s}_3$, which is wider than the subalgebra $\mathfrak{s}_4$. The presented facts demonstrate the importance of using proper megaideals within the framework of the algebraic method for finding point-symmetry groups of systems of differential equations.

Remark 1.30. In contrast to continuous point symmetry transformations, not all discrete point symmetry transformations of (1.1) are extended to ones of its nonlinear Lax representation (1.14). At the same time, the system (1.14) admits, in addition to the expectable point symmetries of simple shifts in v, the discrete point symmetry transformation alternating the sign of v.

Remark 1.31. Although the maximal Lie invariance algebra $\mathfrak{g}$ of the equation (1.1) exhaustively defines the point-symmetry pseudogroup G of this equation, it does not define exhaustively the equation itself. Nevertheless, to single out the equation (1.1) from the entire set of third-order partial differential equations with three independent variables, it suffices to supplement the $\mathfrak{g}$-invariance with a few nice conditions. As such conditions, we have selected admitting the conservation-law characteristics 1, u_{xx} and u_{yy}.

The enhanced description of the point- and contact-symmetry pseudogroups of the dispersionless Nizhnik equation (1.1) is only the first step in further enhancing the results of [92] on Lie reductions of this equation. In Chapter 2, we also reclassify the one-dimensional subalgebras of the algebra $\mathfrak{g}$ and classify its two-dimensional subalgebras, exhaustively carry out Lie reductions of the equation (1.1) and then accurately study its hidden Lie symmetries. In addition, we would like to compute the entire algebra of (local) generalized symmetries of (1.1) and the entire space of its local conservation laws. The above results will create necessary prerequisites for the consideration of nonlocal symmetry-like objects that are related to the equation (1.1).

A similar study can be carried out for both the symmetric and asymmetric Nizhnik equations [94, 124], over the real and the complex fields in the presence of dispersion for the Nizhnik equation in the Novikov–Veselov form and its dispersionless counterpart, as well as for the stationary Nizhnik equation, which was considered in [49, 90] and in [119, Sections 9.7 and 9.8].

Chapter 2

Lie reductions and exact solutions of dispersionless Nizhnik equation

In this chapter, we exhaustively classify the Lie reductions of the real dispersionless Nizhnik equation (1.1) to partial differential equations in two independent variables and to ordinary differential equations.

One-dimensional subalgebras of $\mathfrak{g}_{\rm e}$ (the representation of the contact invariance algebra of the equation (1.1) by vector fields in the evolution form) that are appropriate for Lie reduction were classified in [92] with a minor deficiency. Therein, the corresponding Lie reductions of (1.1) to partial differential equations with two independent variables and further Lie reductions of these equations were performed. Wide families of solutions that are polynomial in (x, y) were constructed as examples of non-invariant solutions. Second-order cosymmetries of the equation (1.1) were found. Since all of them are conservation-law characteristics of this equation, the associated conserved currents were computed as well. At the same time, it was not studied which Lie symmetries of reduced equations are induced by Lie symmetries of the original equation (1.1), and thus a number of presented two-step reductions are in fact needless. Among obtained Lie-invariant solutions of (1.1), there are many equivalent to each other or those containing typos, which makes them incorrect. Careful analysis of reduced ordinary differential equations shows that more of their closed-form solutions can be constructed, and one should take into account the degeneracy of some of these equations.

In this chapter, we correct, enhance and significantly extend results from [92]. We scrupulously carry out each step of the optimized procedure of comprehensive Lie reduction for the dispersionless Nizhnik equation (1.1), which results in finding wide families of new invariant solutions of (1.1) in explicit form in terms of elementary, Lambert and hypergeometric functions as well as in parametric or implicit form.

The first step of the Lie reduction procedure for the equation (1.1) was in fact implemented in [39] and has been reproduced in Sections 1.1 and 1.2 of the present thesis (cf. [92]), where we in particular computed the maximal Lie invariance algebras $\mathfrak{g}$ and $\mathfrak{g}_{\rm L}$ of (1.1) and its nonlinear Lax representation (1.14) as well as their point-symmetry pseudogroups G and $G_{\rm L}$, respectively, and performed a preliminary study of these algebras and pseudogroups.

We compute for the first time point symmetry groups of reduced equations, including their discrete point symmetries, and check whether these symmetries are hidden or induced. Since most of the reduced equations to be considered are quite cumbersome, various versions of the algebraic method by Hydon [60–62] are much more efficient in the course of the above computation than the direct method. In addition, some of the reduced equations of the equation (1.1) are not of maximal rank, and the study of Lie and general point symmetries of differential equations that are not of maximal rank was also carried out for the first time in [127] and reproduced in this chapter. We also make deeper analysis of reduced equations than in most papers in the field of classical group analysis, construct more solutions for more reduced equations and more systematically study hidden symmetries of the original equation. For integrating some of reduced ordinary differential equations for the equation (1.1), we involve the associated Lie reductions of the nonlinear Lax representation (1.14).

In the course of performing the Lie reduction procedure for the equation (1.1), we observe several interesting phenomena. Thus, not all parameters of a family of inequivalent subalgebras are necessarily inherited by the corresponding reduced equations. The limit case for this phenomenon is when all inequivalent subalgebras from a family even parameterized by arbitrary functions correspond, under an appropriate choice of ansatzes, to the same reduced equation. Another display of this phenomenon is the possibility of mapping a class of reduced equations to its proper subclass, which has a less number of parameters. Some equivalent (two-dimensional) subalgebras of the algebra $\mathfrak{g}$ with a nonzero (one-dimensional) intersection induce inequivalent (one-dimensional) subalgebras of the maximal Lie invariance algebra of a reduced partial differential equation obtained by the Lie reduction with respect to the intersection. The algebra $\mathfrak{g}$ is embedded in the algebra $\mathfrak{g}_{\rm L}$ via extending the vector fields from $\mathfrak{g}$ to the dependent variable v of the nonlinear Lax representation (1.14), and thus any Lie reduction of the equation (1.1) has a counterpart among Lie reductions of the system (1.14) but such a counterpart is in general not unique even up to the $G_{\rm L}$-equivalence. In contrast to Lie symmetries, simple and obvious discrete point symmetries of the equation (1.1) induce, even under the optimal choice of ansatzes, complicated and nontrivial discrete point symmetries of the corresponding reduced equations.

For readers' convenience, we mark the constructed solutions of the dispersionless Nizhnik equation (1.1) by the bullet symbol $\bullet$.

The results of Chapter 2 were presented in the paper [127] and in the abstracts of conference talks [4, 6, 7, 126].

2.1. Optimized procedure of Lie reduction

Despite many papers devoted to the construction of exact solutions of systems of partial differential equations using the Lie reduction procedure, the number of papers with correct, proper and systematic studies of Lie reductions for particular important systems modeling real-world phenomena is not as large as it could be expected. Such studies involve cumbersome computations and requires an accurate consideration of many inequivalent cases. Hence a precondition of successfully performing the above procedure is its optimization.

To be specific, we describe the *optimized procedure of Lie reduction* for the case of three independent variables, which is relevant to this chapter. Given a system $\mathcal{L}$ of partial differential equations for unknown functions u in three independent variables, this procedure consists of the following steps; see also further comments after the procedure's description.

1. Compute the maximal Lie invariance (pseudo)algebra $\mathfrak{g}$ and the point symmetry (pseudo)group G of $\mathcal{L}$.
2. Construct complete lists of G-inequivalent one- and two-dimensional subalgebras of $\mathfrak{g}$ and select those among them that are appropriate for using within the framework of Lie reduction.
3. *Lie reductions of codimension one.* For each of the selected one-dimensional subalgebras of $\mathfrak{g}$, say $\mathfrak{s}_1$, find an ansatz for the $\mathfrak{s}_1$-invariant solutions of $\mathcal{L}$ such that the corresponding reduced system $\hat{\mathcal{L}}_1$ of partial differential equations in two independent variables is of the simplest or most convenient form. If the system $\hat{\mathcal{L}}_1$ can be completely integrated or its general solution is expressed in terms of the general solution of a system that has been well studied within the framework of symmetry analysis, then the further consideration of the system $\hat{\mathcal{L}}_1$ and carrying out the Lie reductions of $\mathcal{L}$ with respect to subalgebras of $\mathfrak{g}$ containing, up to G-equivalence, the subalgebra $\mathfrak{s}_1$ are needless.
4. Otherwise, compute the normalizer $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_1)$ of $\mathfrak{s}_1$ in $\mathfrak{g}$, the stabilizer $\mathrm{St}_G(\mathfrak{s}_1)$ of $\mathfrak{s}_1$ in G, the maximal Lie invariance algebra $\hat{\mathfrak{g}}_1$ and the point symmetry group $\hat{G}_1$ of $\hat{\mathcal{L}}_1$ as well as the subalgebra $\tilde{\mathfrak{g}}_1$ of $\hat{\mathfrak{g}}_1$ and the subgroup $\tilde{G}_1$ of $\hat{G}_1$ that are induced by elements of $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_1)$ and $\mathrm{St}_G(\mathfrak{s}_1)$, respectively. Perform the Lie reduction procedure for the system $\hat{\mathcal{L}}_1$ only if $\hat{\mathfrak{g}}_1 \neq \tilde{\mathfrak{g}}_1$ or at least $\hat{G}_1 \neq \tilde{G}_1$, see comments below.

5. *Lie reductions of codimension two.* For each of the two-dimensional subalgebras of $\mathfrak{g}$ that have passed the selection in steps 2 and 3, say $\mathfrak{s}_2$, find an ansatz for the $\mathfrak{s}_2$-invariant solutions of $\mathcal{L}$ such that the corresponding reduced system $\hat{\mathcal{L}}_2$ of ordinary differential equations is of the simplest or most convenient form.
6. Compute the normalizer $\mathrm{N}_\mathfrak{g}(\mathfrak{s}_2)$ of $\mathfrak{s}_2$ in $\mathfrak{g}$, the stabilizer $\mathrm{St}_G(\mathfrak{s}_2)$ of $\mathfrak{s}_2$ in G, the maximal Lie invariance algebra $\hat{\mathfrak{g}}_2$ and the point symmetry group $\hat{G}_2$ of $\hat{\mathcal{L}}_2$ as well as the subalgebra $\tilde{\mathfrak{g}}_2$ of $\hat{\mathfrak{g}}_2$ and the subgroup $\tilde{G}_2$ of $\hat{G}_2$ that are induced by elements of $\mathrm{N}_\mathfrak{g}(\mathfrak{s}_2)$ and $\mathrm{St}_G(\mathfrak{s}_2)$, respectively.
7. Construct, if possible, the general solution of $\hat{\mathcal{L}}_2$ or at least some particular solutions of $\hat{\mathcal{L}}_2$. Use transformations from the group $\tilde{G}_2$ for gauging integration constants in the constructed solutions. Substitute the arranged solutions into the ansatz for the $\mathfrak{s}_2$-invariant solutions, which gives G-inequivalent solutions of the original system $\mathcal{L}$.
8. *Lie reductions of codimension three.* Analyze whether there are Lie reductions of $\mathcal{L}$ with respect to three-dimensional subalgebras of $\mathfrak{g}$ to algebraic equations that lead to new exact solutions of $\mathcal{L}$ in comparison with those constructed in the previous steps using Lie reductions of codimensions one and two. If this is the case, then carry out all such G-inequivalent Lie reductions.

In steps 1, 4 and 6, it is convenient to carry out the computation of the corresponding point symmetry groups by a version of the algebraic method, the automorphism-based version [60–63] (see also further examples, e.g., in [70,123]) or one of the various modifications of the megaideal-based version [27,46,47,85,100] in the case of finite or infinite dimension of the associated maximal Lie invariance algebra, respectively. The direct method [27, 66] may be advantageous for those systems that belong to classes of systems of differential equations with known restrictions for point symmetries of their elements, see [27, 71, 72]. Systems that are not of maximal rank, which are not too uncommon among reduced systems of differential equations,require a specific study, which complicates the consideration, see Section 2.5.2.

Remark 2.1. As a simple illustrative example, consider the quadratic porous medium (Boussinesq) equation $u_t = (uu_x)_x$ for the groundwater pressure u, which describes unsteady flows of groundwater with the presence of a free surface, and its first- and second-level potential equations $v_t = v_x v_{xx}$ and $w_t = \frac{1}{2}(w_{xx})^2$. Each of these equations admits the one-parameter group of shifts with respect to t with the generator ∂_t as its Lie symmetry group. The corresponding invariant solutions are just stationary solutions, and the associated ansatzes $u = \varphi(\omega)$, $v = \psi(\omega)$ and $w = \zeta(\omega)$ with $\omega = x$ respectively reduce these equations to the ordinary differential equations $\varphi\varphi_{\omega\omega} = 0$, $\psi_\omega\psi_{\omega\omega} = 0$ and $(\zeta_{\omega\omega})^2 = 0$, which are not of maximal rank. In particular, the last reduced equation $(\zeta_{\omega\omega})^2 = 0$ is not of maximal rank on the entire set of its solutions.

A subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is appropriate for using within the framework of Lie reduction if and only if satisfies the local transversality condition. In fixed local coordinates, this condition is equivalent to the equality of the ranks of the matrices that are respectively constituted by all the components of basis vector fields of $\mathfrak{s}$ and by solely those corresponding to the independent variables. In step 2, one can classify merely one- and two-dimensional appropriate subalgebras of $\mathfrak{g}$ but, in general, this does not lead to a significant simplification in comparison with the complete classification and the further selection of appropriate subalgebras. Usually, subalgebras of $\mathfrak{g}$ are classified up to their equivalence generated by the group $\mathrm{Inn}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$, which coincides with the $G_{\rm id}$-equivalence, where $G_{\rm id}$ is the identity component of G. At the same time, it is advantageous to use the stronger G-equivalence instead of the $G_{\rm id}$-equivalence since it allows one to reduce the list of subalgebras to be considered. Moreover, this makes the Lie reduction procedure consistent with the natural G-equivalence on the solution set of the system $\mathcal{L}$.

Particular attention in steps 3 and 5 should be paid to the optimal choice of ansatzes [53,54,105,118]. Given a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, an $\mathfrak{s}$-invariant ansatz is defined up to an arbitrary point transformation of invariant variables. In other words, there is an infinite family of $\mathfrak{s}$-invariant ansatzes, and selecting a proper representative in this family usually leads to an essential simplification of the corresponding reduced system and its further study. The simplicity of the form of reduced systems and its certain similarity to the form of the original system $\mathcal{L}$ do not exhaust possible criteria in the course of selecting ansatzes. Another criterion is to unify the form of reduced systems for a subset of listed families of G-inequivalent subalgebras of $\mathfrak{g}$ for embedding them into a nice superclass of differential equations and unifying their study. After reducing the system $\mathcal{L}$ using a preliminary ansatz, one can improve the form of the obtained reduced system by a point transformation of invariant variables and then optimize the ansatz by means of combining it with this transformation. At the same time, such transformations may significantly complicate the form of ansatzes. To preserve the balance between the simplicity of ansatzes and the simplicity of the corresponding reduced systems, sometimes it is necessary to transform ansatzes only partially.

Elements of optimal lists of subalgebras of $\mathfrak{g}$ can in general be not only single subalgebras but also families of subalgebras parameterized by arbitrary constants or, if the algebra $\mathfrak{g}$ is infinite-dimensional, even by arbitrary functions. Lie reductions of the system $\mathcal{L}$ with respect to subalgebras from such a family result in a class $\mathcal{C}$ of reduced systems with subalgebra parameters as its arbitrary elements rather than in a collection of single reduced systems. Thus, the study of Lie symmetries for systems from the class $\mathcal{C}$ should be realized as the solution to the group classification problem for this class.

We would like to emphasize that further Lie reductions of a reduced system of partial differential equations with two independent variables in step 3 should be carried out only if this system admits point symmetries that are not induced by point symmetries[2.1] of the original system $\mathcal{L}$ and thus called *hidden point symmetries*of $\mathcal{L}$ associated with the codimension-one reduction under consideration. See the description of the optimized procedure of step-by-step reductions with involving hidden symmetries in [71, Section B].

Remark 2.2. In this context, the term *hidden symmetries* was first used in [130]. Other terms for this notion in the literature are *additional* [96, Example 3.5], *non-induced* [53, 54] or *Type-II hidden* [14, 15] symmetries. The first example of such symmetries was given in [64] but become known after its discussion in [96, Example 3.5]. A systematic study of them is rather seldom and has been carried out only for a few famous systems of differential equations, in particular, for the Navier–Stokes equations describing flows of an incompressible viscous fluid [53, 54], the (1+1)-dimensional generalized Burgers equations $u_t + uu_x + f(t, x)u_{xx} = 0$ [105], the two-dimensional degenerate Burgers equation $u_t + uu_x - u_{yy} = 0$ [123], the Boiti–Leon–Pempinelli system [85], the (1+2)-dimensional ultraparabolic Fokker–Planck equation $u_t + xu_y = u_{xx}$ [123] as well as the dispersionless Nizhnik equation in the present thesis. Interesting particular examples of hidden symmetries of several hydrodynamic models were presented in [21, Chapter 1].

The study of Lie and general point symmetries of the derived reduced systems, identifying hidden symmetries of $\mathcal{L}$ among them and using such hidden symmetries for further Lie reduction of the corresponding reduced systems is a necessary part of the comprehensive Lie reduction procedure.

It is useless and counterproductive to consider the other step-by-step Lie reductions, whose second steps are based on induced symmetries of reduced systems. There are at least two sources of inconveniences in the course

[2.1] The induction of Lie symmetries of a reduced system by Lie symmetries of the original system was first discussed in [103, Section 20.4].

of such reductions, which implicitly lead to the consideration of multiple essentially equivalent reductions.

To make the first source evident, consider a particular case, where the system $\mathcal{L}$ admits two commuting Lie-symmetry vector fields Q^1 and Q^2 such that the subalgebras $\mathfrak{s}_1^\mu := \langle Q^1 + \mu Q^2\rangle$ of $\mathfrak{g}$ parameterized by an arbitrary constant μ are pairwise G-inequivalent and each of them satisfies the local transversality condition and is thus appropriate for Lie reduction of $\mathcal{L}$. It is obvious that for any μ, the vector field Q^2 belongs to the normalizer of $\mathfrak{s}_1^\mu$ in $\mathfrak{g}$. Therefore, it induces a Lie-symmetry vector field $\hat{Q}^{2,\mu}$ of the reduced system $\mathcal{L}_1^\mu$ for $\mathfrak{s}_1^\mu$-invariant solutions of $\mathcal{L}$. Suppose that the algebra $\langle\hat{Q}^{2,\mu}\rangle$ also satisfies the local transversality condition. Thus, we have the infinite family of two-step reductions, where for each μ, the system $\mathcal{L}$ is first reduced to the system $\mathcal{L}_1^\mu$ using the algebra $\mathfrak{s}_1^\mu$ and then the system $\mathcal{L}_1^\mu$ is further reduced using the algebra $\langle\hat{Q}^{2,\mu}\rangle$. Moreover, the first steps of these reductions are definitely not equivalent to each other. Nevertheless, each of these two-step reductions is equivalent to the same one-step Lie reduction of the system $\mathcal{L}$ with respect to the two-dimensional subalgebra $\langle Q^1, Q^2\rangle$ of $\mathfrak{g}$. For invariance algebras of more complicated structure, equivalences between multi-step reductions are in general not so obvious, and establishing them requires an additional analysis.

The second source is that $\hat{G}_1$-inequivalent (one-dimensional) subalgebras of the maximal Lie invariance algebra $\mathfrak{a}_1$ of a reduced system $\hat{\mathcal{L}}_1$ of partial differential equations may correspond to equivalent (two-dimensional) subalgebras of $\mathfrak{g}$; recall that by $\hat{G}_1$ we denote the point symmetry group of $\hat{\mathcal{L}}_1$, see this and other related notations in the above description of the optimized procedure of Lie reduction. More specifically, suppose that the system $\hat{\mathcal{L}}_1$ is obtained by the Lie reduction of the original system $\mathcal{L}$ with respect to a one-dimensional subalgebra $\mathfrak{s}_1 = \langle Q^0\rangle$ of $\mathfrak{g}$, and vector fields Q^1 and Q^2 belong to the normalizer $\mathrm{N}_\mathfrak{g}(\mathfrak{s}_1)$ of $\mathfrak{s}_1$ in $\mathfrak{g}$, thus inducing elements $\hat{Q}^1$ and $\hat{Q}^2$ of $\mathfrak{a}_1$. In addition, suppose that the subalgebras $\langle Q^0, Q^1\rangle$

and $\langle Q^0, Q^2\rangle$ are G-equivalent and the equivalence is established only by a transformation $\Phi \in G$ that does not belong to the stabilizer $\mathrm{St}_G(\mathfrak{s}_1)$ of $\mathfrak{s}_1$ in G. Then the transformation Φ does not induce a point symmetry of $\hat{\mathcal{L}}_1$, and thus the subalgebras $\langle \hat{Q}^1\rangle$ and $\langle \hat{Q}^2\rangle$ of $\mathfrak{a}_1$ are in general $\hat{G}_1$-inequivalent. In terms of reductions and solutions, this means that the inequivalent two-step Lie reductions with the first step using the subalgebra $\mathfrak{s}_1$ of $\mathfrak{g}$ and the second step using the subalgebras $\langle \hat{Q}^1\rangle$ and $\langle \hat{Q}^2\rangle$ of $\mathfrak{a}_1$ result in the G-equivalent families of the $\langle Q^0, Q^1\rangle$- and the $\langle Q^0, Q^2\rangle$-invariant solutions, respectively. See Remark 2.8 for a nontrivial example of the described situation, which arises in the course of studying Lie reductions of the dispersionless Nizhnik equation (1.1).

This is why the best strategy is to completely avoid multi-step reductions not involving hidden symmetries.

We do not include the classification of three-dimensional subalgebras of $\mathfrak{g}$ in step 2 since in general, it is a much more complicated problem than those for dimensions one and two and it is not required for the Lie reduction procedure in its entity. Only a small number of three-dimensional subalgebras satisfy the selection criterion from step 8, if they exist at all. This is why it is better to merely classify the selected subalgebras directly in step 8. For example, the maximal Lie invariance algebra of the dispersionless Nizhnik equation (1.1) contains no three-dimensional subalgebras that are appropriate to step 8. In a similar way, we may also consider two-dimensional subalgebras of $\mathfrak{g}$ but the reached simplification is not essential in comparison with their complete classification.

The system $\mathcal{L}$ can possess families of trivial or obvious solutions that can be easily guessed without applying Lie reduction or other methods. Moreover, these families can contain solutions that are invariant with respect to subalgebras of $\mathfrak{g}$ whose dimensions are greater than or equal to the number of independent variables, and thus such solutions can repeatedly arise in the course of performing the Lie reduction procedure for the system $\mathcal{L}$. It is beneficial to find these families of solutions before step 3 and exclude their elements under the further listing of solutions. Similar solution families can be constructed in step 3 and should be treated analogously. Section 2.3, the solution family (2.4) and the treatment of trivial solutions in Section 2.5 illustrate the above remark.

In addition to classical integration methods, a number of other techniques can be applied to finding exact solutions of a reduced system $\mathcal{R}$ of ordinary differential equations. An obvious approach is to use Lie symmetries of the system $\mathcal{R}$ for at least lowering its order, see Section 2.5.2. One can try to construct first integrals of $\mathcal{R}$ by means of the direct method [18,19] supposing a certain ansatz for the associated integrating multipliers, see [85, Section 6] for the application of this technique to reduced systems of ordinary differential equations for the Boiti–Leon–Pempinelli system. One can also look for objects that are related to the original system $\mathcal{L}$ within the framework of symmetry analysis of differential equations and induce analogous objects for the system $\mathcal{R}$. These objects include not only Lie and general point symmetries, first integrals and integrating multipliers but also Lagrangian and Hamiltonian structures and (linear and nonlinear) Lax representations. Induced objects can then be involved in obtaining exact solutions of $\mathcal{R}$. See, e.g., Section 2.5.2 for using induced nonlinear Lax representations.

2.2. Classification of one- and two-dimensional subalgebras

To carry out Lie reductions of codimension one and two for the equation (1.1) in the optimal way, we should classify one- and two-dimensional subalgebras of the algebra $\mathfrak{g}$ up to G_*-equivalence. Instead of the classical approach for finding inner automorphisms [96, Section 3.3], we act on $\mathfrak{g}$ by G via pushing forward of vector fields by elements of G. Recall that this way is more convenient for computing in the infinite-dimensional case [25,45]. Moreover, it also allows us to properly use the entire complete point-symmetry group G and not be limited to its connected component of the identity transformation. Thus the non-identity adjoint actions of elementary transformations from G on vector fields spanning $\mathfrak{g}$ are merely

$$
\begin{aligned}
&\mathcal{D}^t_*(T)D^t(\tau)=D^t\big(\hat T_t^{-1}\tau(\hat T)\big),\\
&\mathcal{D}^t_*(T)P^x(\chi)=P^x\big(\hat T_t^{-1/3}\chi(\hat T)\big),\quad \mathcal{D}^t_*(T)P^y(\rho)=P^y\big(\hat T_t^{-1/3}\rho(\hat T)\big),\\
&\mathcal{D}^t_*(T)R^x(\alpha)=R^x\big(\hat T_t^{1/3}\alpha(\hat T)\big),\quad \mathcal{D}^t_*(T)R^y(\beta)=R^y\big(\hat T_t^{1/3}\beta(\hat T)\big),\\
&\mathcal{D}^t_*(T)Z(\sigma)=Z\big(\sigma(\hat T)\big),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{D}^{\rm s}_*(C)P^x(\chi)=P^x(C\chi),\quad \mathcal{D}^{\rm s}_*(C)P^y(\rho)=P^y(C\rho),\\
&\mathcal{D}^{\rm s}_*(C)R^x(\alpha)=R^x(C^2\alpha),\quad \mathcal{D}^{\rm s}_*(C)R^y(\beta)=R^y(C^2\beta),\\
&\mathcal{D}^{\rm s}_*(C)Z(\sigma)=Z(C^3\sigma),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}^x_*(X^0)D^t(\tau)=D^t(\tau)&+P^x\big(\tau X^0_t-\tfrac13\tau_tX^0\big)\\
&+\tfrac12R^x\big(X^0(\tau X^0_t)_t-\tfrac13\tau_{tt}(X^0)^2-\tau(X^0_t)^2\big)\\
&-\tfrac16Z\big((X^0)^2(\tau X^0_t)_t-\tfrac13\tau_{tt}(X^0)^3-\tau X^0(X^0_t)^2\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}^y_*(Y^0)D^t(\tau)=D^t(\tau)&+P^y(\tau Y^0_t-\tfrac13\tau_tY^0)\\
&+\tfrac12R^y\big(Y^0(\tau Y^0_t)_t-\tfrac13\tau_{tt}(Y^0)^2-\tau(Y^0_t)^2\big)\\
&-\tfrac16Z\big((Y^0)^2(\tau Y^0_t)_t-\tfrac13\tau_{tt}(Y^0)^3-\tau Y^0(Y^0_t)^2\big),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{R}^x_*(W^1)D^t(\tau)=D^t(\tau)+R^x(\tau W^1_t+\tfrac13\tau_tW^1),\\
&\mathcal{R}^y_*(W^2)D^t(\tau)=D^t(\tau)+R^y(\tau W^2_t+\tfrac13\tau_tW^2),\\
&\mathcal{Z}_*(W^0)D^t(\tau)=D^t(\tau)+Z(\tau W^0_t),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{P}^x_*(X^0)D^{\rm s}=D^{\rm s}-P^x(X^0),\quad \mathcal{R}^x_*(W^1)D^{\rm s}=D^{\rm s}-2R^x(W^1),\\
&\mathcal{P}^y_*(Y^0)D^{\rm s}=D^{\rm s}-P^y(Y^0),\quad \mathcal{R}^y_*(W^2)D^{\rm s}=D^{\rm s}-2R^y(W^2),\\
&\mathcal{Z}_*(W^0)D^{\rm s}=D^{\rm s}-3Z(W^0),
\end{aligned}
$$

$$\mathcal{P}^x_*(X^0)P^x(\chi)=P^x(\chi)+R^x\big(\chi_tX^0-\chi X^0_t\big)-\tfrac12 Z\big(\chi_t(X^0)^2-\chi X^0X^0_t\big),$$
$$\mathcal{P}^x_*(X^0)R^x(\alpha)=R^x(\alpha)-Z(\alpha X^0),$$
$$\mathcal{R}^x_*(W^1)P^x(\chi)=P^x(\chi)+Z(\chi W^1),$$

$$\mathcal{P}^y_*(Y^0)P^y(\rho)=P^y(\rho)+R^y(\rho_tY^0-\rho Y^0_t)-\tfrac12 Z(\rho_t(Y^0)^2-\rho Y^0_tY^0),$$
$$\mathcal{P}^y_*(Y^0)R^y(\beta)=R^y(\beta)-Z(\beta Y^0),$$
$$\mathcal{R}^y_*(W^2)P^y(\rho)=P^y(\rho)+Z(\rho W^2),$$

$$\mathcal{J}_*P^x(\chi)=P^y(\chi),\quad \mathcal{J}_*P^y(\rho)=P^x(\rho),$$
$$\mathcal{J}_*R^x(\alpha)=R^y(\alpha),\quad \mathcal{J}_*R^y(\beta)=R^x(\beta),$$

where $\hat T$ is the inverse of the function T. At the same time, a part of adjoint actions can be computed via mimicking the classical approach if the corresponding Lie series has a finite number of nonzero terms.

Lemma 2.3. *A complete list of G-inequivalent one-dimensional subalgebras of the algebra $\mathfrak g$ is exhausted by the following subalgebra families:*

$$\mathfrak s^\delta_{1.1}=\big\langle D^t(1)+\delta D^{\rm s}\big\rangle,\quad \mathfrak s_{1.2}=\big\langle D^{\rm s}\big\rangle,$$
$$\mathfrak s^\rho_{1.3}=\big\langle P^x(1)+P^y(\rho)\big\rangle,\quad \mathfrak s^\beta_{1.4}=\big\langle P^x(1)+R^y(\beta)\big\rangle,$$
$$\mathfrak s^\beta_{1.5}=\big\langle R^x(1)+R^y(\beta)\big\rangle,\quad \mathfrak s_{1.6}=\big\langle Z(t)\big\rangle,\quad \mathfrak s_{1.7}=\big\langle Z(1)\big\rangle,$$

where $\delta\in\{0,1\}\pmod G$, and ρ and β run through the set of smooth functions of t with $\rho\neq0$.

Proof. Let $\mathfrak s_1=\langle Q\rangle$ be a one-dimensional subalgebra of $\mathfrak g$ spanned by a nonvanishing vector field $Q=D^t(\tau)+\lambda D^{\rm s}+P^x(\chi)+P^y(\rho)+R^x(\alpha)+R^y(\beta)+Z(\sigma)$ from $\mathfrak g$. Here τ, χ, ρ, α, β and σ are arbitrary smooth functions of t and λ is an arbitrary constant that do not simultaneously vanish.

If the function τ is nonzero, then we use $\mathcal D^t_*(T)$ with $T_t=1/\tau$ to set $\tau=1$ and, preserving the notation of the parameter functions, successively act on the (currently modified) vector field Q by $\mathcal P^x_*(X^0)\circ\mathcal P^y_*(Y^0)$ with $X^0_t-\lambda X^0=-\chi$ and $Y^0_t-\lambda Y^0=-\rho$ to set $\chi=\rho=0$, by $\mathcal R^x_*(W^1)\circ\mathcal R^y_*(W^2)$

with $W^1_t-2\lambda W^1=-\alpha$ and $W^2_t-2\lambda W^2=-\beta$ to set $\alpha=\beta=0$ and by $\mathcal Z_*(W^0)$ with $W^0_t-3\lambda W^0=-\sigma$ to set $\sigma=0$. Thus, we obtain $Q=D^t(1)+\lambda D^{\rm s}$. If $\lambda\neq0$, we can set $\lambda=1$ by simultaneously scaling t and the entire Q and, if necessary, alternating their signs. In other words, the subalgebra $\mathfrak s_1$ with $\tau\neq0$ is G-equivalent to a one in the family $\{\mathfrak s^0_{1.1},\,\mathfrak s^1_{1.1}\}$.

Suppose that $\tau=0$ and $\lambda\neq0$. Changing the basis element Q, we first set $\lambda=1$. Then, preserving the notation of the parameter functions and successively act on the (currently modified) vector field Q by $\mathcal P^x_*(\chi)\circ\mathcal P^y_*(\rho)$ to set $\chi=\rho=0$, by $\mathcal R^x_*(\frac12\alpha)\circ\mathcal R^y_*(\frac12\beta)$ to set $\alpha=\beta=0$ and by $\mathcal Z_*(\frac13\sigma)$ to set $\sigma=0$, which leads to the subalgebra $\mathfrak s_{1.2}$.

Let $\tau=0$, $\lambda=0$ and $\chi\rho\neq0$. Analogously to the above cases, a chain of simplifying successive actions is $\mathcal D^t_*(T)$ with $T_t=\chi^{-3}$, $\mathcal P^x_*(X^0)\circ\mathcal P^y_*(Y^0)$ with $X^0_t=\alpha$ and $\rho Y^0_t-\rho_tY^0=\beta$ and by $\mathcal R^x_*(W^1)$ with $W^1=-\sigma$, which gives $\chi=1$, $\alpha=\beta=0$ and $\sigma=0$. Thus, we have the subalgebra $\mathfrak s^\rho_{1.3}$.

Let $\tau=0$ and $\lambda=0$ and exactly one of the parameter functions χ and ρ is nonzero. Up to the permutation of x and y, we can assume without loss of generality that $\chi\neq0$ and $\rho=0$. Similarly to the previous case, we set $\chi=1$, $\alpha=0$ and $\sigma=0$, and obtain the subalgebra $\mathfrak s^\beta_{1.4}$.

Further we assume $\tau=0$, $\lambda=0$ and $\chi=\rho=0$.

If $(\alpha,\beta)\neq(0,0)$, then due to the possibility of permuting x and y, we can assume, without loss of generality, $\alpha\neq0$ and set $\alpha=1$ and $\sigma=0$ modulo the G-equivalence, which gives the subalgebra $\mathfrak s^\beta_{1.5}$.

Otherwise, $\alpha=\beta=0$ and $\sigma\neq0$. The consideration splits into two cases $\sigma_t\neq0$ and $\sigma_t=0$, where the subalgebra $\mathfrak s_1$ is G-equivalent to $\mathfrak s_{1.6}$ and $\mathfrak s_{1.7}$, respectively. □

Lemma 2.4. *A complete list of G-inequivalent two-dimensional subalgebras of the algebra $\mathfrak g$ is exhausted by the non-abelian algebras*

$$\mathfrak s^\lambda_{2.1}=\big\langle D^t(1),\,D^t(t)+\lambda D^{\rm s}\big\rangle,$$
$$\mathfrak s^\nu_{2.2}=\big\langle D^t(1),\,D^t(t)-\tfrac13D^{\rm s}+P^x(1)+P^y(\nu)\big\rangle,$$

$$\begin{aligned}
&\mathfrak{s}_{2.3}^{\nu} = \big\langle D^t(1),\ D^t(t) + \tfrac{1}{6}D^{\rm s} + R^x(1) + R^y(\nu)\big\rangle,\\
&\mathfrak{s}_{2.4} = \big\langle D^t(1),\ D^t(t) + Z(1)\big\rangle,\\
&\mathfrak{s}_{2.5}^{\lambda\mu} = \big\langle D^t(1) + \lambda D^{\rm s},\ P^x({\rm e}^{(\lambda-1)t}) + \mu P^y({\rm e}^{(\lambda-1)t})\big\rangle,\\
&\mathfrak{s}_{2.6}^{\lambda\delta} = \big\langle D^t(1) + \lambda D^{\rm s},\ P^x({\rm e}^{(\lambda-1)t}) + \delta R^y({\rm e}^{(2\lambda-1)t})\big\rangle,\\
&\mathfrak{s}_{2.7}^{\lambda\nu} = \big\langle D^t(1) + \lambda D^{\rm s},\ R^x({\rm e}^{(2\lambda-1)t}) + \nu R^y({\rm e}^{(2\lambda-1)t})\big\rangle,\\
&\mathfrak{s}_{2.8}^{\lambda} = \big\langle D^t(1) + \lambda D^{\rm s},\ Z({\rm e}^{(3\lambda-1)t})\big\rangle,\\
&\mathfrak{s}_{2.9}^{\tilde\rho} = \big\langle D^{\rm s},\ P^x(1) + P^y(\tilde\rho)\big\rangle,\\
&\mathfrak{s}_{2.10}^{\beta} = \big\langle D^{\rm s},\ R^x(1) + R^y(\beta)\big\rangle,\\
&\mathfrak{s}_{2.11} = \big\langle D^{\rm s},\ Z(t)\big\rangle, \quad \mathfrak{s}_{2.12} = \big\langle D^{\rm s},\ Z(1)\big\rangle,
\end{aligned}$$

and the abelian algebras

$$\begin{aligned}
&\mathfrak{s}_{2.13} = \big\langle D^t(1),\ D^{\rm s}\big\rangle,\\
&\mathfrak{s}_{2.14}^{\delta\nu\delta'} = \big\langle D^t(1) + \delta D^{\rm s},\ P^x({\rm e}^{\delta t}) + \nu P^y({\rm e}^{\delta t}) + \delta' R^y({\rm e}^{2\delta t})\big\rangle,\\
&\mathfrak{s}_{2.15}^{\delta\nu} = \big\langle D^t(1) + \delta D^{\rm s},\ R^x({\rm e}^{2\delta t}) + \nu R^y({\rm e}^{2\delta t})\big\rangle,\\
&\mathfrak{s}_{2.16}^{\delta} = \big\langle D^t(1) + \delta D^{\rm s},\ Z({\rm e}^{3\delta t})\big\rangle,\\
&\mathfrak{s}_{2.17}^{\rho\alpha\beta} = \big\langle P^x(1) + R^y(\beta),\ P^y(\rho) + R^x(\rho\beta)\big\rangle,\\
&\mathfrak{s}_{2.18}^{\rho\beta\sigma} = \big\langle P^x(1) + P^y(\rho),\ -R^x(\rho\beta) + R^y(\beta) + Z(\sigma)\big\rangle_{(\beta,\sigma)\neq(0,0)},\\
&\mathfrak{s}_{2.19}^{\beta^1\beta^2} = \big\langle P^x(1) + R^y(\beta^1),\ R^y(\beta^2)\big\rangle_{\beta^2\neq 0},\\
&\mathfrak{s}_{2.20}^{\beta\sigma} = \big\langle P^x(1) + R^y(\beta),\ Z(\sigma)\big\rangle_{\sigma\neq 0},\\
&\mathfrak{s}_{2.21}^{\alpha\beta^1\beta^2} = \big\langle R^x(1) + R^y(\beta^1),\ R^x(\alpha) + R^y(\beta^2)\big\rangle_{\beta^2\neq\alpha\beta^1},\\
&\mathfrak{s}_{2.22}^{\alpha\beta\sigma} = \big\langle R^x(1) + R^y(\beta),\ R^x(\alpha) + R^y(\alpha\beta) + Z(\sigma)\big\rangle_{\alpha_t\neq 0},\\
&\mathfrak{s}_{2.23}^{\beta\sigma} = \big\langle R^x(1) + R^y(\beta),\ Z(\sigma)\big\rangle_{\sigma\neq 0}, \quad \mathfrak{s}_{2.24}^{\sigma} = \big\langle Z(t),\ Z(\sigma)\big\rangle_{\sigma_{tt}\neq 0},\\
&\mathfrak{s}_{2.25}^{\sigma} = \big\langle Z(1),\ Z(\sigma)\big\rangle_{\sigma_t\neq 0},
\end{aligned}$$

where ρ, $\tilde\rho$, α, β, β^1, β^2 and σ run through the set of smooth functions of t with $\rho\neq 0$, $\lambda\in\mathbb{R}$, $\mu\in[-1,1]\setminus\{0\}$ and $\nu\in[-1,1]$ $(\mathrm{mod}\ G)$, $\delta,\delta'\in\{0,1\}$ $(\mathrm{mod}\ G)$, and the conditions indicated after the corresponding subalgebras should be satisfied as well.

Proof. Consider a two-dimensional subalgebra $\mathfrak{s}_2 = \langle Q^1, Q^2\rangle$ of $\mathfrak{g}$ spanned by two (linearly independent) vector fields

$$Q^i = D^t(\tau^i) + \lambda^i D^{\mathrm{s}} + P^x(\chi^i) + P^y(\rho^i) + R^x(\alpha^i) + R^y(\beta^i) + Z(\sigma^i),$$

from $\mathfrak{g}$ with arbitrary smooth functions τ^i, χ^i, ρ^i, α^i, β^i and σ^i of t and arbitrary constants λ^i such that the tuples $(\tau^i, \lambda^i, \chi^i, \rho^i, \alpha^i, \beta^i, \sigma^i)$, $i = 1, 2$, are linearly independent. Moreover, since $[Q^1, Q^2] \in \langle Q^1, Q^2\rangle$, up to changing the basis (Q^1, Q^2), we can assume that either $[Q^1, Q^2] = Q^1$ or $[Q^1, Q^2] = 0$ if the subalgebra $\mathfrak{s}_2$ is non-abelian or abelian, respectively. Consider these cases separately. For each of the obtained families of subalgebras, we do not indicate a final tuning of its basis elements, which involves permuting or scaling these elements or omitting superfluous indices.

I. The commutation relation $[Q^1, Q^2] = Q^1$ implies $\lambda^1 = 0$.

First suppose that the functions τ^1 and τ^2 are linearly independent. Then the projections $\tau^i\partial_t$ of Q^i, $i = 1, 2$ on the t-line span a two-dimensional Lie algebra of vector fields on the real line with $[\tau^1\partial_t, \tau^2\partial_t] = \tau^1\partial_t$. In view of the classical Lie theorem, there exists a point transformation $\tilde t = T(t)$ of t that pushes forward the vector fields $\tau^1\partial_t$ and $\tau^2\partial_t$ to $\partial_{\tilde t}$ and $\tilde t\partial_{\tilde t}$. This means that the action by $\mathcal{D}^t_*(T)$ allows us to set $\tau^1 = 1$ and $\tau^2 = t$. Following the first case of the proof of Lemma 2.3, we can further set χ^1, ρ^1, α^1, β^1 and σ^1 to 0. Re-denote $(\lambda^2, \chi^2, \rho^2, \alpha^2, \beta^2, \sigma^2)$ as $(\lambda, \chi, \rho, \alpha, \beta, \sigma)$. Under the derived constraints, the commutation relation $[Q^1, Q^2] = Q^1$ is equivalent to the equations $\chi_t = \rho_t = \alpha_t = \beta_t = \sigma_t = 0$, i.e., all these subalgebra parameters are constants. The pushforward by a transformation Φ from G does not change the vector field $Q^1 = D^t(1)$ up to its scaling if and only if $T = at + b$ for some constants a and b and the parameter functions X^0, Y^0, W^0, W^1 and W^2 are constants, whereas the constant C is not additionally constrained,

$$\begin{aligned}\Phi\colon\quad &\tilde t = at + b, \quad \tilde x = Cx + X^0, \quad \tilde y = Cy + Y^0,\\ &\tilde u = C^3u + W^1x + W^2y + W^0,\end{aligned}$$

that is,

$$\Phi=\mathcal{D}^t_*(at{+}b)\circ\mathcal{P}^x_*(X^0)\circ\mathcal{P}^y_*(Y^0)\circ\mathcal{R}^x_*(W^1/C)\circ\mathcal{R}^y_*(W^2/C)\circ\mathcal{Z}_*(W^0)\circ\mathcal{D}^{\rm s}_*(C).$$

Pushing forward $\mathfrak{s}_2$ by such Φ with $a=C=1$ in addition, we have $\Phi_*Q^1=Q^1$ and

$$\begin{aligned}\Phi_*Q^2=Q^2-bQ^1-\left(\lambda+\frac13\right)\big(P^x(X^0)+P^y(Y^0)\big)\\
-\left(2\lambda-\frac13\right)\big(R^x(W^1)+R^y(W^2)\big)\\
-Z\big(\alpha X^0+\beta Y^0-\chi W^1-\rho W^2\\
-\left(2\lambda-\frac13\right)(W^1X^0+W^2Y^0)+3\lambda W^0\big).\end{aligned}$$

This implies that for general values of λ, we can set $\chi=\rho=\alpha=\beta=\sigma=0$, obtaining the subalgebra family $\{\mathfrak{s}^\lambda_{2.1}\}$. Subalgebras of the considered kind that are not G-equivalent to elements of the family $\{\mathfrak{s}^\lambda_{2.1}\}$ correspond to the special values $-\frac13$, $\frac16$ and 0 of λ, where in addition $(\chi,\rho)\neq(0,0)$, $(\alpha,\beta)\neq(0,0)$ and $\sigma\neq0$, respectively. In each of these cases, the other parameters in the tuple $(\chi,\rho,\alpha,\beta,\sigma)$ can be set to zero by Φ_* as above. Using the permutation of x and y, we replace the inequality $(\chi,\rho)\neq(0,0)$ by $\chi\neq0$ and $|\rho|\leqslant|\chi|$ and the inequality $(\alpha,\beta)\neq(0,0)$ by $\alpha\neq0$ and $|\beta|\leqslant|\alpha|$. Acting by $\mathcal{D}^t_*(at)$ or $\mathcal{D}^{\rm s}_*(C)$, we scale the nonzero parameter among χ, α or σ to 1, which leads to the subalgebras $\mathfrak{s}^\rho_{2.2}$ with $|\rho|\leqslant1$, $\mathfrak{s}^\beta_{2.3}$ with $|\beta|\leqslant1$ or $\mathfrak{s}_{2.4}$, respectively.

Now, let the functions τ^1 and τ^2 are linearly dependent but not simultaneously zero. The commutation relation $[Q^1,Q^2]=Q^1$ implies that $\tau^1=0$ and $\tau^2\neq0$ in this case. Following the first case of the proof of Lemma 2.3, we can set $Q^2=D^t(1)+\lambda D^{\rm s}$, where we re-denote λ^2 by λ. Recalling again the commutation relation $[Q^1,Q^2]=Q^1$, we obtain $\chi^1=\nu_1{\rm e}^{(\lambda-1)t}$, $\rho^1=\nu_2{\rm e}^{(\lambda-1)t}$, $\alpha^1=\nu_3{\rm e}^{(2\lambda-1)t}$, $\beta^1=\nu_4{\rm e}^{(2\lambda-1)t}$ and $\sigma^1=\nu_5{\rm e}^{(3\lambda-1)t}$ with constants ν_1, …, ν_5. For further simplification, we can apply only the pushforwards that do not change the form of the basis element Q^2 up to its linearly combining with Q^1. The pushforward by an element Φ of G has this property if and only if the transformation Φ or $\mathscr{J}\circ\Phi$ is of the form (1.4) with

$$
\begin{aligned}
&T=t+b,\quad X^0=\kappa_1\mathrm{e}^{\lambda t}-\kappa_0C\nu_1\mathrm{e}^{(\lambda-1)t},\quad Y^0=\kappa_2\mathrm{e}^{\lambda t}-\kappa_0C\nu_2\mathrm{e}^{(\lambda-1)t},\\
&W^1=\kappa_3\mathrm{e}^{2\lambda t}-(\kappa_0C^3\nu_3-\lambda\kappa_0C^2\nu_1\kappa_1)\mathrm{e}^{(2\lambda-1)t}-\frac{\lambda-1}{2}\kappa_0^2C^3\nu_1^2\mathrm{e}^{2(\lambda-1)t},\\
&W^2=\kappa_4\mathrm{e}^{2\lambda t}-(\kappa_0C^3\nu_4-\lambda\kappa_0C^2\nu_2\kappa_2)\mathrm{e}^{(2\lambda-1)t}-\frac{\lambda-1}{2}\kappa_0^2C^3\nu_2^2\mathrm{e}^{2(\lambda-1)t},\\
&W^0=\kappa_5\mathrm{e}^{3\lambda t}-\kappa_0(C^3\nu_5+\nu_1\kappa_3+\nu_2\kappa_4)\mathrm{e}^{(3\lambda-1)t}\\
&\qquad+\frac12\kappa_0^2C^3(\nu_1\nu_3+\nu_2\nu_4)\mathrm{e}^{(3\lambda-2)t}-\frac{\lambda}{2}\kappa_0^2C^2(\nu_1^2\kappa_1+\nu_2^2\kappa_2)\mathrm{e}^{(3\lambda-2)t}\\
&\qquad+\frac{\lambda-1}{6}\kappa_0^3C^3(\nu_1^3+\nu_2^3)\mathrm{e}^{3(\lambda-1)t},
\end{aligned}
$$

where C, b and $\kappa_0,\dots,\kappa_5$ are arbitrary constants with $C\neq0$. In addition, we can multiply Q^1 by an arbitrary nonzero constant. As a result, we set

- $\nu_1=1,\ |\nu_2|\leqslant1,\ \nu_3=\nu_4=\nu_5=0$ if $\nu_1\nu_2\neq0$,
- $\nu_1=1,\ \nu_2=0,\ \nu_3=\nu_5=0,\ \nu_4\in\{0,1\}$ if $\nu_1\nu_2=0,\ (\nu_1,\nu_2)\neq(0,0)$,
- $\nu_3=1,\ |\nu_4|\leqslant1,\ \nu_5=0$ if $\nu_1=\nu_2=0,\ (\nu_3,\nu_4)\neq(0,0)$,
- $\nu_5=1$ otherwise.

After re-denoting the respective parameters, this corresponds to the subalgebras $\mathfrak{s}_{2.5}^{\lambda\mu}$, $\mathfrak{s}_{2.6}^{\lambda\delta}$, $\mathfrak{s}_{2.7}^{\lambda\nu}$ and $\mathfrak{s}_{2.8}^{\lambda}$.

If $\tau^1=\tau^2=0$, then $\lambda^2\neq0$ and we follow the second case of the proof of Lemma 2.3 and set $\chi^2=\rho^2=\alpha^2=\beta^2=\sigma^2=0$, which gives $Q^2=\lambda^2D^{\mathrm{s}}$. The commutation relation $[Q^1,Q^2]=Q^1$ implies that there are three possible cases, $\lambda^2=1$ and $\alpha^1=\beta^1=\sigma^1=0$, $\lambda^2=\frac12$ and $\chi^1=\rho^1=\sigma^1=0$ or $\lambda^2=\frac13$ and $\chi^1=\rho^1=\alpha^1=\beta^1=0$. Permuting x and y if necessary and acting by $\mathcal{D}_*^t(T)$ with an appropriate value of the parameter function T, we can set $\chi^1=1$, $\alpha^1=1$ or $\sigma^1\in\{t,1\}$ in the first, the second or the third cases, which leads to the subalgebras $\mathfrak{s}_{2.9}^{\rho^1}$, $\mathfrak{s}_{2.10}^{\beta^1}$ or $\mathfrak{s}_{2.11}$ and $\mathfrak{s}_{2.12}$, respectively.

II. Suppose that the subalgebra $\mathfrak{s}_2$ is abelian, $[Q^1,Q^2]=0$. Then the functions τ^1 and τ^2 are necessarily linearly dependent.

Let in addition the tuples (τ^1,λ^1) and (τ^2,λ^2) are linearly independent. Linearly combining Q^1 and Q^2, we can set $\tau^1\neq 0$, $\lambda^1=0$, $\tau^2=0$ and $\lambda^2\neq 0$. The successive action by $\mathcal{D}^t_*(T)$ with $T_t=1/\tau^1$ and the new value of τ^1 allows us to set $\tau^1=1$. Following the second case of the proof of Lemma 2.3, we set $\chi^2=\rho^2=\alpha^2=\beta^2=\sigma^2=0$. Then the commutation relation $[Q^1,Q^2]=0$ implies $\chi^1=\rho^1=\alpha^1=\beta^1=\sigma^1=0$, and thus we have the subalgebra $\mathfrak{s}_{2.13}$

If the tuples (τ^1,λ^1) and (τ^2,λ^2) are linearly dependent and the functions τ^1 and τ^2 do not simultaneously vanish, then we linearly combine Q^1 and Q^2 to set $\tau^1\neq 0$, $\tau^2=0$ and $\lambda^2=0$. According to the first case of the proof of Lemma 2.3, we can reduce Q^1 to the form $D^t(1)+\lambda^1D^{\rm s}$. In view of the commutation relation $[Q^1,Q^2]=0$, the parameter functions in Q^2 are $\chi^2=\nu_1{\rm e}^{\lambda t}$, $\rho^2=\nu_2{\rm e}^{\lambda t}$, $\alpha^2=\nu_3{\rm e}^{2\lambda t}$, $\beta^2=\nu_4{\rm e}^{2\lambda t}$ and $\sigma^2=\nu_5{\rm e}^{3\lambda t}$ with constants ν_1, …, ν_5. The pushforward by an element Φ of G, which is necessarily of the form (1.4), does not change the form of the basis element Q^1 up to its linearly combining with Q^2 if and only if the parameters of Φ have the following form:

$$
\begin{aligned}
&T=t+b,\quad X^0=(\kappa_1+\kappa_0C\nu_1t){\rm e}^{\lambda t},\quad Y^0=(\kappa_2+\kappa_0C\nu_2t){\rm e}^{\lambda t},\\
&W^1=\left(\kappa_3+\kappa_0C^3\nu_3t-\lambda\kappa_0C^2\nu_1\kappa_1t-\kappa_0^2C^3\nu_1^2t-\frac{\lambda}{2}\kappa_0^2C^3\nu_1^2t^2\right){\rm e}^{2\lambda t},\\
&W^2=\left(\kappa_4+\kappa_0C^3\nu_4t-\lambda\kappa_0C^2\nu_2\kappa_2t-\kappa_0^2C^3\nu_2^2t-\frac{\lambda}{2}\kappa_0^2C^3\nu_2^2t^2\right){\rm e}^{2\lambda t},\\
&W^0=\Bigg(\kappa_5+\kappa_0(C^3\nu_5+\nu_1\kappa_3+\nu_2\kappa_4)t+\frac{1}{2}\kappa_0^2C^3(\nu_1\nu_3+\nu_2\nu_4)t^2\\
&\qquad-\frac{\lambda}{2}\kappa_0^2C^2(\nu_1^2\kappa_1+\nu_2^2\kappa_2)t^2-\frac{1}{2}\kappa_0^3C^3(\nu_1^3+\nu_2^3)t^2\\
&\qquad-\frac{\lambda}{6}\kappa_0^3C^3(\nu_1^3+\nu_2^3)t^3\Bigg){\rm e}^{3\lambda t},
\end{aligned}
$$

where C, b and κ_0, …, κ_5 are arbitrary constants with $C\neq 0$. We can also push forward $\mathfrak{s}_2$ by $\mathcal{J}$ or multiply Q^2 by an arbitrary nonzero constant. Hence we can set

- $\nu_1=1,\ |\nu_2|\leqslant 1,\ \nu_3=0,\ \nu_4\in\{0,1\},\ \nu_5=0$ if $(\nu_1,\nu_2)\neq 0$,
- $\nu_3=1,\ |\nu_4|\leqslant 1,\ \nu_5=0$ if $\nu_1=\nu_2=0,\ (\nu_3,\nu_4)\neq 0$,
- $\nu_5=1$ otherwise,

which corresponds, up to re-denoting parameters, to the subalgebras $\mathfrak{s}_{2.14}^{\delta\nu\delta'}$, $\mathfrak{s}_{2.15}^{\delta\nu}$ and $\mathfrak{s}_{2.16}^{\delta}$.

If $\tau^1=\tau^2=0$, then it follows from the commutation relation $[Q^1,Q^2]=0$ that also $\lambda^1=\lambda^2=0$ since otherwise the vector fields Q^1 and Q^2 are linearly dependent, and $\chi^1\chi^2_t-\chi^1_t\chi^2=0$, $\rho^1\rho^2_t-\rho^1_t\rho^2=0$, $\chi^1\alpha^2-\chi^2\alpha^1+\rho^1\beta^2-\rho^2\beta^1=0$, i.e., the parameter functions χ^1 and χ^2 (resp. ρ^1 and ρ^2) are linearly dependent. Suppose that the tuples (χ^1,ρ^1) and (χ^2,ρ^2) are linearly independent. Linearly combining Q^1 and Q^2, we make $\chi^1\rho^2\neq 0$ and $\chi^2=\rho^1=0$. Then the action by $\mathcal{D}^t_*(T)$ with $T_t=(\chi^1)^{-3}$ allows us to set $\chi^1=1$, after which $\alpha^2=\rho^2\beta^1$, and we obtain the subalgebra $\mathfrak{s}_{2.17}^{\rho\alpha\beta}$. If the tuples (χ^1,ρ^1) and (χ^2,ρ^2) are linearly dependent but not simultaneously zero, then we linearly combine Q^1 and Q^2 and, if necessary, permute x and y to make $\chi^1\neq 0$ and $\chi^2=\rho^2=0$. Successively acting by $\mathcal{D}^t_*(T)$ with $T_t=(\chi^1)^{-3}$, by $\mathcal{P}^x_*(X^0)$ with $X^0_t=\alpha^1$ and by $\mathcal{R}^x_*(W^1)$ with $W^1=-\sigma^1$, we set $\chi^1=1$, $\alpha^1=0$ and $\sigma^1=0$, which results in $\alpha^2=-\rho^1\beta^2$. The further simplifications are $\beta^1=0$ if $\rho^1\neq 0$, $\sigma^2=0$ if $\rho^1=0$ and $\beta^2\neq 0$, and no meaningful simplification is possible if $\rho^1=\beta^2=0$. This gives the subalgebras $\mathfrak{s}_{2.18}^{\rho\beta\sigma}$, $\mathfrak{s}_{2.19}^{\beta^1\beta^2}$ and $\mathfrak{s}_{2.20}^{\beta\sigma}$, respectively. In the case $\chi^1=\rho^1=\chi^2=\rho^2=0$, the consideration splits according to the additional conditions that

- $\alpha^1\beta^2\neq\alpha^2\beta^1$,
- $\alpha^1\beta^2=\alpha^2\beta^1$ but the tuples (α^1,β^1) and (α^2,β^2) are linearly independent,
- the tuples (α^1,β^1) and (α^2,β^2) are linearly dependent but not simultaneously zero,
- $\alpha^1=\beta^1=\alpha^2=\beta^2=0$, and σ^1_t and σ^2_t are linearly independent,

- ◦ $\alpha^1 = \beta^1 = \alpha^2 = \beta^2 = 0$, and σ^1_t and σ^2_t are linearly dependent,

and after obvious simplifications, we obtain the subalgebras $\mathfrak{s}^{\alpha\beta^1\beta^2}_{2.21}$, $\mathfrak{s}^{\alpha\beta\sigma}_{2.22}$, $\mathfrak{s}^{\beta\sigma}_{2.23}$, $\mathfrak{s}^{\sigma}_{2.24}$ and $\mathfrak{s}^{\sigma}_{2.25}$. □

Remark 2.5. The statements of Lemmas 2.3 and 2.4 should be interpreted in the following way. Subalgebras from different families or within each of the families parameterized only by constants are definitely G-inequivalent. At the same time, there is an inessential equivalence between subalgebras within each of the families parameterized by functions that does not allow us to further simplify the general form of subalgebras from the family. For example, subalgebras $\mathfrak{s}^{\rho}_{1.3}$ and $\mathfrak{s}^{\tilde\rho}_{1.3}$ are G-inequivalent if and only if $\tilde\rho(t) = \rho(at+b)$ for some $a, b \in \mathbb{R}$ or $\tilde\rho = (\rho(\hat T))^{-1}$, where $\hat T$ is the inverse of a solution T of the equation $T_t = c\rho^{-3}$ for some $c \in \mathbb{R}$.

Remark 2.6. For the purpose of Lie reduction of the equation (1.1) to differential equations with less number of independent variables, it would suffice to only classify one-dimensional subalgebras of rank one and two-dimensional subalgebras of rank two. Nevertheless, we decided to present the respective complete classifications since they require not much more effort than the above partial classifications do. Moreover, this is instructive given the fact that the number of correct classifications of subalgebras of Lie algebras (especially infinite-dimensional ones) in the literature is not great.

Due to the analogy of the structures of $(\mathfrak{g}, G)$ and $(\mathfrak{g}_{\rm L}, G_{\rm L})$, we can easily obtain the classifications of one- and two-dimensional subalgebras of the algebra $\mathfrak{g}_{\rm L}$ using Lemmas 2.3 and 2.4, respectively. Here we consider only one-dimensional subalgebras of $\mathfrak{g}_{\rm L}$.

Lemma 2.7. *A complete list of $G_{\rm L}$-inequivalent one-dimensional subalgebras of the algebra $\mathfrak{g}_{\rm L}$ is exhausted by the following algebras:*

$$\bar{\mathfrak{s}}^{\delta\delta'}_{1.1} = \big\langle \bar D^t(1) + \delta \bar D^{\rm s} + \delta' \bar P^v \big\rangle_{\delta\delta'=0}, \quad \bar{\mathfrak{s}}_{1.2} = \big\langle \bar D^{\rm s} \big\rangle,$$
$$\bar{\mathfrak{s}}^{\rho\delta}_{1.3} = \big\langle \bar P^x(1) + \bar P^y(\rho) + \delta \bar P^v \big\rangle, \quad \bar{\mathfrak{s}}^{\beta\delta}_{1.4} = \big\langle \bar P^x(1) + \bar R^y(\beta) + \delta \bar P^v \big\rangle,$$

$$\bar{\mathfrak{s}}^{\beta\delta}_{1.5} = \langle \bar{R}^x(1) + \bar{R}^y(\beta) + \delta\bar{P}^v\rangle, \quad \bar{\mathfrak{s}}^{\delta}_{1.6} = \langle \bar{Z}(t) + \delta\bar{P}^v\rangle,$$
$$\bar{\mathfrak{s}}^{\delta}_{1.7} = \langle \bar{Z}(1) + \delta\bar{P}^v\rangle, \quad \bar{\mathfrak{s}}_{1.8} = \langle \bar{P}^v\rangle,$$

where $\delta, \delta' \in \{0, 1\}$, and ρ and β run through the set of smooth functions of t with $\rho \neq 0$.

2.3. Trivial solutions

It is obvious that the equation (1.1) is identically satisfied on the solution set of the differential constraint $u_{xy} = 0$ or, equivalently, on the set of functions of (t, x, y) with additive separation of the variables x and y. In other words, the equation (1.1) has the solutions of the form

- $u = w(t, x) + \tilde{w}(t, y),$ (2.1)

where w and $\tilde{w}$ are sufficiently smooth functions of their arguments. Calling these solutions trivial is justified by the fact that the equation (1.1) is a potential equation for the dispersionless Nizhnik system $p_t = (h^1p)_x + (h^2p)_y$, $h^1_y = p_x$, $h^2_x = p_y$ with the relation $p = u_{xy}$, $h^1 = u_{xx}$, $h^2 = u_{yy}$, and thus a solution is of the form (2.1) for the equation (1.1) if and only if it corresponds to a solution of the dispersionless Nizhnik system with zero principal component p.

Within the family (2.1), there is the subfamily of solutions satisfying the differential constraints $u_{xy} = u_{xxxx} = u_{yyyy} = 0$ and $u_{xxx} = u_{yyy}$ and thus having the form

$$\begin{aligned} u = {} & W^5(t)(x^3 + y^3) + W^3(t)x^2 + W^4(t)y^2 \\ & + W^1(t)x + W^2(t)y + W^0(t), \end{aligned} \tag{2.2}$$

where the coefficients W^0, …, W^5 are arbitrary sufficiently smooth functions of t. The solutions from the subfamily (2.2) are even more trivial than general elements of the family (2.1) since each solution of the form (2.2) is G-equivalent to the constant zero solution $u = 0$.

The above trivial solutions of the equation (1.1) often arise in the course of its Lie reductions. Identifying such solutions among constructed ones and neglecting them in addition to listing solutions up to the G-equivalence allow us to better arrange the found families of invariant solutions. Note that modulo the G-equivalence, we can arbitrarily change or neglect summands of the form $W^1(t)x + W^2(t)y + W^0(t)$ in any solution of (1.1).

2.4. Lie reductions of codimension one

Among subalgebras listed in Lemma 2.3, only subalgebras $\mathfrak{s}^{\delta}_{1.1}$, $\mathfrak{s}_{1.2}$, $\mathfrak{s}^{\rho}_{1.3}$ and $\mathfrak{s}^{\beta}_{1.4}$ are appropriate to be used for Lie reduction of the equation (1.1). We collect G-inequivalent codimension-one Lie reductions of the equation (1.1) in Table 2.1. There, for each of the above one-dimensional subalgebras of $\mathfrak{g}$, we present a constructed ansatz for u, the corresponding reduced partial differential equation in two independent variables, where $w = w(z_1, z_2)$ is the new unknown function of the invariant independent variables (z_1, z_2). The subscripts 1 and 2 of w denote the differentiation with respect to z_1 and z_2, respectively.

Table 2.1. G-inequivalent Lie reductions with respect to one-dimensional subalgebras of $\mathfrak{g}$.

$\subset \mathfrak{g}$	u	z_1	z_2	Reduced equation
$\mathfrak{s}^{\delta}_{1.1}$	$\mathrm{e}^{3\delta t}w - \frac{1}{6}\delta(x^3+y^3)$	$\mathrm{e}^{-\delta t}x$	$\mathrm{e}^{-\delta t}y$	$(w_{11}w_{12})_1 + (w_{12}w_{22})_2 = 3\delta w_{12}$
$\mathfrak{s}_{1.2}$	x^3w	t	y/x	$(z_2w_{22} - 2w_2)_1 = \big((z_2w_{22} - 2w_2)w_{22}\big)_2 - (z_2\partial_2 - 2)\big((z_2w_{22} - 2w_2)(z_2^2w_{22} - 4z_2w_2 + 6w)\big)$
$\mathfrak{s}^{\rho}_{1.3}$	$w - \frac{1}{6}\rho_t\rho^{-1}y^3$	t	$\rho^{-1}y - x$	$w_{122} + 2(1-\rho^{-3})w_{22}w_{222} = 0$
$\mathfrak{s}^{\beta}_{1.4}$	$w + \beta xy$	t	y	$\beta w_{222} = \beta_1$

We study each of the listed reduced equations separately, indexing it by the number of the corresponding one-dimensional subalgebra of $\mathfrak{g}$.

1.1. $\mathfrak{s}^{\delta}_{1.1}=\big\langle D^t(1)+\delta D^{\rm s}\big\rangle$, $\delta\in\{0,1\}$ (mod G). The maximal Lie invariance algebra of reduced equation 1.1^δ is[2.2]

$$\mathfrak{a}^0_{1.1}=\langle D^z,\, w\partial_w,\, \partial_{z_1},\, \partial_{z_2},\, z_1\partial_w,\, z_2\partial_w,\, \partial_w\rangle \quad\text{if}\quad \delta=0,$$
$$\mathfrak{a}^1_{1.1}=\langle \tilde D^z,\, \partial_{z_1},\, \partial_{z_2},\, z_1\partial_w,\, z_2\partial_w,\, \partial_w\rangle \quad\text{if}\quad \delta=1.$$

Here $D^z:=z_1\partial_{z_1}+z_2\partial_{z_2}$ and $\tilde D^z:=z_1\partial_{z_1}+z_2\partial_{z_2}+3w\partial_w$. All their elements are induced by Lie symmetries of the original equation (1.1). Indeed, the normalizer of the subalgebra $\mathfrak{s}^\delta_{1.1}$ in $\mathfrak{g}$ is

$$\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}^0_{1.1})=\langle D^t(1), D^t(t), D^{\rm s}, P^x(1), P^y(1), R^x(1), R^y(1), Z(1)\rangle,$$
$$\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}^1_{1.1})=\langle D^t(1), D^{\rm s}, P^x({\rm e}^t), P^y({\rm e}^t), R^x({\rm e}^{2t}), R^y({\rm e}^{2t}), Z({\rm e}^{3t})\rangle$$

for $\delta=0$ and $\delta=1$, respectively (see a similar computation in [123, Section 3]). The Lie-symmetry vector fields $D^t(1)+\delta D^{\rm s}$, $D^{\rm s}$, $P^x({\rm e}^{\delta t})$, $P^y({\rm e}^{\delta t})$, $R^x({\rm e}^{2\delta t})$, $R^y({\rm e}^{2\delta t})$, $Z({\rm e}^{3\delta t})$ and, for $\delta=0$, $3D^t(t)$ of the equation (1.1) induce the Lie-symmetry vector fields 0, $\tilde D^z$, ∂_{z_1}, ∂_{z_2}, $z_1\partial_w$, $z_2\partial_w$, ∂_w and, for $\delta=0$, D^z of reduced equation 1.1, respectively.

Therefore, any two-step Lie reduction of the equation (1.1) to an ordinary differential equation, where the first step is reduction 1.1 and the second step is a Lie reduction of reduced equation 1.1, is equivalent to a direct Lie reduction to an ordinary differential equation using a two-dimensional subalgebra of $\mathfrak{g}$. This means that there is no need to carry out Lie reductions of reduced equation 1.1.

For each $\delta\in\{0,1\}$, let us compute the point-symmetry group $G^\delta_{1.1}$ of the reduced equation 1.1^δ by the algebraic method. Up to the antisymmetry of the Lie bracket, the nonzero commutation relations between the basis

[2.2] In contrast to reduced equation 1.1^1, its counterpart with $\delta=0$ loses the property of maximal rank on the submanifold $\mathcal{M}_0$ of the manifold $\mathcal{M}$. Here the manifold $\mathcal{M}$ is defined by this equation in the jet space ${\rm J}^3(\mathbb{R}^2_{z_1,z_2}\times\mathbb{R}_w)$ and the submanifold $\mathcal{M}_0$ is singled out in $\mathcal{M}$ by the consistent system $w_{11}=w_{12}=w_{22}=0$, $w_{112}=w_{122}=w_{111}+w_{222}=0$. It can be proved by the classical infinitesimal method that the maximal Lie invariance algebra of the complement $\mathcal{M}\setminus\mathcal{M}_0$ of $\mathcal{M}_0$ in $\mathcal{M}$ coincides with the algebra $\mathfrak{a}^0_{1.1}$. At the same time, the submanifold $\mathcal{M}_0$ is also invariant with respect to this algebra. Therefore, the maximal Lie invariance algebra of reduced equation 1.1^0 is indeed the algebra $\mathfrak{a}^0_{1.1}$.

vector fields of the algebra $\mathfrak{a} := \mathfrak{a}^{\delta}_{1.1}$ are exhausted by

$$[D^z, \partial_{z_1}] = -\partial_{z_1}, \quad [D^z, \partial_{z_2}] = -\partial_{z_2},$$
$$[D^z, z_1\partial_w] = z_1\partial_w, \quad [D^z, z_2\partial_w] = z_2\partial_w,$$
$$[w\partial_w, z_1\partial_w] = -z_1\partial_w, \quad [w\partial_w, z_2\partial_w] = -z_2\partial_w, \quad [w\partial_w, \partial_w] = -\partial_w,$$
$$[\partial_{z_1}, z_1\partial_w] = \partial_w, \quad [\partial_{z_2}, z_2\partial_w] = \partial_w,$$

and

$$[\tilde D^z, \partial_{z_1}] = -\partial_{z_1}, \quad [\tilde D^z, \partial_{z_2}] = -\partial_{z_2},$$
$$[\tilde D^z, z_1\partial_w] = -2z_1\partial_w, \quad [\tilde D^z, z_2\partial_w] = -2z_2\partial_w,$$
$$[\tilde D^z, \partial_w] = -3\partial_w, \quad [\partial_{z_1}, z_1\partial_w] = \partial_w, \quad [\partial_{z_2}, z_2\partial_w] = \partial_w$$

if $\delta = 0$ and $\delta = 1$, respectively. We first find megaideals of the algebra $\mathfrak{a}$ applying techniques that do not require the knowledge of the automorphism group $\mathrm{Aut}(\mathfrak{a})$ [26, 111]. Then we use the constructed megaideals for simplifying the computation of $\mathrm{Aut}(\mathfrak{a})$ and obtain the remaining megaideals. As a result, the complete list of proper megaideals of $\mathfrak{a}$ is as follows:

$$\mathfrak{m}_1 := \mathfrak{a}' = \big\langle \partial_{z_1},\, \partial_{z_2},\, z_1\partial_w,\, z_2\partial_w,\, \partial_w \big\rangle, \quad \mathfrak{m}_2 := \mathfrak{a}'' = \mathfrak{z}(\mathfrak{m}_1) = \big\langle \partial_w \big\rangle,$$
$$\mathfrak{m}_3 := \mathrm{C}_{\mathfrak{a}}(\mathfrak{m}_2) = \big\langle D^z \big\rangle \dotplus \mathfrak{m}_1, \quad \mathfrak{m}_4 := \big\langle D^z + 2w\partial_w \big\rangle \dotplus \mathfrak{m}_1 \quad \text{if} \quad \delta = 0,$$
$$\mathfrak{m}_3 := \big\langle z_1\partial_w,\, z_2\partial_w,\, \partial_w \big\rangle, \quad \mathfrak{m}_4 := \big\langle \partial_{z_1},\, \partial_{z_2},\, \partial_w \big\rangle \quad \text{if} \quad \delta = 1.$$

Denote $\mathfrak{m}_0 := \mathfrak{a}$. Let a point transformation Φ: $(\tilde z_1, \tilde z_2, \tilde w) = (Z^1, Z^2, W)$ in the space with the coordinates (z_1, z_2, w), where (Z^1, Z^2, W) is a tuple of smooth functions of (z_1, z_2, w) with nonvanishing Jacobian, preserve the equation (1.1). Necessary conditions for this are $\Phi_*\mathfrak{m}_k \subseteq \mathfrak{m}_k$, $k = 0, \dots, 4$. Hereafter the indices i and j run from 1 to 2, and we assume summation with respect to repeated indices. The conditions $\Phi_*\partial_w \in \mathfrak{m}_2$ and $\Phi_*(z_i\partial_w) \in \mathfrak{m}_1$ imply that

$$Z^i = a_{ij}z_j + a_{i0}, \quad W = cw + W^0(z_1, z_2),$$

where a_{ij}, a_{i0} and c are constants with $c\det(a_{ij}) \neq 0$, and W^0 is a smooth function of (z_1, z_2). Then the conditions $\Phi_*\partial_{z_i} \in \mathfrak{m}_4$ if $\delta = 1$ or $\Phi_*\partial_{z_i} \in \mathfrak{m}_1$

and $\Phi_* D^z \in \mathfrak{m}_3$ if $\delta = 0$ in addition give that $W^0 = b_i z_i + b_0$ for some constants b_i and b_0. Since no further constraints on Φ can be derived within the framework of the algebraic method, we continue the computation with the direct method, obtaining $a_{11} = a_{22}$, $a_{12} = a_{21}$, $a_{11}a_{12} = 0$, $(a_{11}, a_{12}) \neq (0, 0)$ and, if $\delta = 1$, $c = a^3$, where a is the nonzero value among a_{11} and a_{12}. This means that there are exactly two independent, up to composing with each other and with continuous point symmetry transformations of the equation (1.1), discrete point symmetry transformations of this equation. They are the involutions, the one that permutes the independent variables z_1 and z_2, $(\tilde z_1, \tilde z_2, \tilde w) = (z_2, z_1, w)$, and the one that simultaneously alternates the signs of all the variables, $(\tilde z_1, \tilde z_2, \tilde w) = (-z_1, -z_2, -w)$. These transformations are respectively induced by the discrete point symmetries $\mathscr{J}$ and $\mathscr{I}^{\rm s}$ of the original equation (1.1). The fact of inducing all Lie symmetries of reduced equations 1.1^δ follows from that for the corresponding Lie invariance algebras $\mathfrak{a}^\delta_{1.1}$. Therefore, for each $\delta \in \{0, 1\}$ the group $G^\delta_{1.1}$ is entirely induced by the stabilizer of $\mathfrak{s}^\delta_{1.1}$ in G.

The subalgebra $\mathfrak{s}^\delta_{1.1}$ of $\mathfrak{g}$ is associated with the subalgebra(s) $\bar{\mathfrak{s}}^{\delta\delta'}_{1.1}$ of $\mathfrak{g}_{\rm L}$ with $\delta\delta' = 0$. The extension of ansatz 1.1 to v is $v = {\rm e}^{\frac32\delta t} q + \delta' t$. Here and in the next case, $q = q(z_1, z_2)$ is the invariant unknown function that replaces v, and, as for w, the subscripts 1 and 2 of q denote the differentiation with respect to z_1 and z_2, respectively. The corresponding family of reduced systems for the nonlinear Lax representation (1.14) is associated with the subalgebra family $\bar{\mathfrak{s}}^{\delta\delta'}_{1.1}$ from Lemma 2.7 and consists of the systems

$$\frac13(q_1^3 + q_2^3) + q_1 w_{11} + q_2 w_{22} = \frac32\delta q + \delta', \quad w_{12} = -q_1 q_2, \tag{2.3}$$

each of which can be interpreted as a nonlinear Lax representation for reduced equation 1.1 with the same value of δ, cf. the introductive part of [92, Section 4.1]. In other words, reduced equation 1.1 is the compatibility condition of the system (2.3) with respect to q. Note that for $\delta = 0$ we thus construct two inequivalent nonlinear Lax representations, with $\delta' = 0$ and with $\delta' = 1$.

Reduced equation 1.1^0 is just the stationary dispersionless Nizhnik equation. Note that its counterpart with dispersion was studied in [90].

Remark 2.8. A complete list of $G^1_{1.1}$-inequivalent one-dimensional subalgebras of the algebra $\mathfrak{a}^1_{1.1}$ is exhausted by the subalgebras

$$\mathfrak{b}_1=\langle\tilde D^z\rangle,\quad \mathfrak{b}_2^{\nu\kappa\varsigma}=\langle\partial_{z_1}+\nu\partial_{z_2}+\kappa z_1\partial_w+\varsigma z_2\partial_w\rangle,$$
$$\mathfrak{b}_3^{\nu}=\langle z_1\partial_w+\nu z_2\partial_w\rangle,\quad \mathfrak{b}_4=\langle\partial_w\rangle,$$

where $\nu\in[-1,1]$, and $\varsigma\neq0$ if $\nu\in\{-1,1\}$ and $(\kappa,\varsigma)\neq(0,0)$, cf. [92, Proposition 2]. A further gauging of subalgebra parameters is possible only within the second family $\{\mathfrak{b}_2^{\nu\kappa\varsigma}\}$, where one of the parameters κ or ς, if nonzero, can be set to be equal to 1 up to the $G^1_{1.1}$-equivalence. At the same time, the subalgebra $\mathfrak{b}_2^{\nu\kappa\varsigma}$ is induced by the subalgebra

$$\breve{\mathfrak{b}}_2^{\nu\kappa\varsigma}=\langle D^t(1)+\delta D^{\rm s},\ P^x(e^{\delta t})+\nu P^y(e^{\delta t})+\kappa R^x(e^{2\delta t})+\varsigma R^y(e^{2\delta t})\rangle \text{ of } \mathfrak{g},$$

which is G-equivalent to the subalgebra $\mathfrak{s}_{2.14}^{\delta\nu\delta'}$, where $\delta'=0$ if $\kappa=\varsigma$ and $\delta'=1$ otherwise. In other words, if $\tilde\nu=\nu$, the tuples (κ,ς) and $(\tilde\kappa,\tilde\varsigma)$ are not proportional with a nonzero multipliers and $\kappa-\varsigma$ and $\tilde\kappa-\tilde\varsigma$ are simultaneously either are equal to zero or are not, then the subalgebras $\mathfrak{b}_2^{\nu\kappa\varsigma}$ and $\mathfrak{b}_2^{\tilde\nu\tilde\kappa\tilde\varsigma}$ of $\mathfrak{a}^1_{1.1}$ are $G^1_{1.1}$-inequivalent, whereas the associated subalgebras $\breve{\mathfrak{b}}_2^{\nu\kappa\varsigma}$ and $\breve{\mathfrak{b}}_2^{\tilde\nu\tilde\kappa\tilde\varsigma}$ are G-equivalent. This is why the inequivalent two-step reductions, where the first step is reduction 1.1^1 and the second step is based on subalgebras $\mathfrak{b}_2^{\nu\kappa\varsigma}$ and $\mathfrak{b}_2^{\tilde\nu\tilde\kappa\tilde\varsigma}$ of $\mathfrak{a}^1_{1.1}$ with the above constraints on the subalgebra parameters, definitely results in G-equivalent families of invariant solutions of the dispersionless Nizhnik equation (1.1), cf. [92, Section 4.1.1.2]. The above phenomenon has not been described in the literature.

1.2. $\mathfrak{s}_{1.2}=\big\langle D^{\rm s}\big\rangle$. The same conclusion on the superfluousness of two-step Lie reduction can be made for reduced equation 1.2. Its maximal Lie invariance algebra is

$$\mathfrak{a}_{1.2}=\big\langle\breve D(\tau)\big\rangle\quad\text{with}\quad \breve D(\tau):=\tau\partial_{z_1}-\left(\tau_1 w+\frac{1}{18}\tau_{11}(z_2^3+1)\right)\partial_w.$$

Here and in what follows the parameter function τ runs through the set of smooth functions of z_1. It is obvious that the normalizer of the subalgebra $\mathfrak{s}_{1.2}$ in $\mathfrak{g}$ is $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_{1.2}) = \langle D^t(\tau), D^{\mathrm{s}}\rangle$. The Lie-symmetry vector field $D^t(\tau)$ of the equation (1.1) induces the element of $\mathfrak{a}_{1.2}$ with the same value of the parameter function τ, whereas D^{s} is mapped to 0. In other words, the entire maximal Lie invariance algebra $\mathfrak{a}_{1.2}$ of reduced equation 1.2 is induced by $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_{1.2})$. Therefore, similarly to reduced equation 1.1, further Lie reductions of reduced equation 1.2 are needless.

In fact, all point symmetries of reduced equation 1.2 are induced by point symmetries of the original equation (1.1). To show this, we compute the point-symmetry group of reduced equation 1.2 using the most general version of the algebraic method. Again, consider a point transformation Φ: $(\tilde z_1, \tilde z_2, \tilde w) = (Z^1, Z^2, W)$ in the space with the coordinates (z_1, z_2, w), where (Z^1, Z^2, W) is a tuple of smooth functions of (z_1, z_2, w) with nonvanishing Jacobian. The algebra $\mathfrak{a} := \mathfrak{a}_{1.2}$ is infinite-dimensional and contains no proper megaideals. This is why the only convenient necessary condition for the transformation Φ to preserve the equation (1.1) is $\Phi_*\mathfrak{a} \subseteq \mathfrak{a}$, which expands to $\Phi_*\breve D(\tau) = \breve D(\tilde\tau)$. Componentwise splitting the latter condition with each of the specific values $\tau = z_1^i$, $i = 0, \dots, 3$, $\Phi_*\breve D(z_1^i) = \breve D(\tilde\tau^i)$, and recombining the derived determining equations for the components of Φ, we in particular obtain the equation

$$\tilde\tau^3 - 3z_1\tilde\tau^2 + 3z_1^2\tilde\tau^1 - z_1^3\tilde\tau^0 = 0.$$

Since at most one function among $\tilde\tau^i$, $i = 0, \dots, 3$, can be constant, this equation can be solved with respect to Z^1, giving $Z^1 = Z^1(z_1)$. It is obvious that reduced equation 1.2 admits the discrete point symmetry $\breve{\mathcal{J}}^1$: $(\tilde z_1, \tilde z_2, \tilde w) = (-z_1, z_2, -w)$, which is induced by the discrete point symmetry $\mathcal{J}^{\mathrm{i}} \circ \mathcal{J}^{\mathrm{s}}$ of the equation (1.1). Up to factoring out the transformation $\breve{\mathcal{J}}^1$ and Lie symmetries of reduced equation 1.2, each of which is also induced, we can assume that $Z^1 = z_1$. For this restricted form of Φ, we have $\Phi_*\breve D(\tau) = \breve D(\tau)$. Splitting this condition componentwise and with respect to the parameter function τ and its derivatives τ_{z_1} and $\tau_{z_1z_1}$ leads to the equations $Z^2_{z_1} = Z^2_w = 0$, $W_{z_1} = 0$, $wW_w = W$ and $(z_2^3+1)W_w = (Z^2)^3+1$. Therefore, $Z^2 = Z^2(z_2)$ and $W = W^1(z_2)w$ with conditions $Z^2_{z_2} \neq 0$ and $W^1 := \big((Z^2)^3+1\big)/(z_2^3+1)$. To derive more constraints on Φ, we should continue the computation with the direct method. This only gives two solutions, $Z^2 = z_2$ and $Z^2 = 1/z_2$, which correspond to the identity transformation and the discrete point symmetry $\breve{\mathscr{J}}$: $(\tilde z_1, \tilde z_2, \tilde w) = (z_1, z_2^{-1}, z_2^{-3}w)$. The transformation $\breve{\mathscr{J}}$ is induced by the discrete point symmetry $\mathscr{J}$ of the equation (1.1). Therefore, the entire point-symmetry group of reduced equation 1.2 is induced by the stabilizer of $\mathfrak{s}_{1.2}$ in G.

The associated subalgebra of $\mathfrak{g}_{\rm L}$ for the subalgebra $\mathfrak{s}_{1.2}$ of $\mathfrak{g}$ is $\bar{\mathfrak{s}}_{1.2}$. The v-component of the extension of ansatz 1.2 is $v = |x|^{3/2}q$. The corresponding reduced system for the nonlinear Lax representation (1.14) is

$$
\begin{aligned}
q_1 = {}& \frac{\varepsilon}{3}\left(\frac32 q - z_2q_2\right)^3 + \frac{\varepsilon}{3}q_2^3 \\
& + (z_2^2w_{22} - 4z_2w_2 + 6w)\left(\frac32 q - z_2q_2\right) + w_{22}q_2, \\
\varepsilon q_2 & \left(\frac32 q - z_2q_2\right) = z_2w_{22} - 2w_2
\end{aligned}
$$

with $\varepsilon = \operatorname{sgn} x$, which can be interpreted, after solving with respect to (q_1, q_2), as a nonlinear Lax representation for reduced equation 1.2, cf. [92, Section 4.2] up to typos.

1.3. $\mathfrak{s}^{\rho}_{1.3} = \big\langle P^x(1) + P^y(\rho)\big\rangle$ with $\rho = \rho(t) \neq 0$.

If $\rho \equiv 1$, then reduced equation 1.3 degenerates to $w_{122} = 0$, and its general solution is $w = f(z_2) + \varrho^1(z_1)z_2 + \varrho^0(z_1)$, where ϱ^0 and ϱ^1 are arbitrary functions of $z_1 = t$, and f is an arbitrary sufficiently smooth function of $z_2 = y - x$, cf. [92, Eq. (60)]. Up to the G-equivalence, the coefficients ϱ^0 and ϱ^1 can be assumed to vanish. The corresponding family of solutions of (1.1) is

- $u = f(y - x)$. $\quad$ (2.4)

For any ρ with $\rho \not\equiv 1$, meaning that $\rho \not\equiv 1$ on each open interval of the domain of ρ, we can use the change of independent variables $\tilde z_1 = 2\int(1-\rho^{-3})\,\mathrm{d}z_1$, $\tilde z_2 = z_2$ for modifying ansatz 1.3^ρ and reduced equation 1.3^ρ to the form

$$u = w(\tilde z_1, \tilde z_2) - \frac{\rho_t}{6\rho}y^3, \quad \tilde z_1 = 2\int \frac{\rho^3-1}{\rho^3}\mathrm{d}t, \quad \tilde z_2 = \frac{y}{\rho} - x,$$
$$w_{\tilde z_1\tilde z_2\tilde z_2} + w_{\tilde z_2\tilde z_2}w_{\tilde z_2\tilde z_2\tilde z_2} = 0. \tag{2.5}$$

Thus, the class of reduced equations 1.3^ρ associated with the subalgebra family $\{\mathfrak{s}^\rho_{1.3} \mid \rho \not\equiv 1\}$ collapses to the unary class, whose single element is the equation (2.5). In other words, the G-inequivalent subalgebras $\mathfrak{s}^\rho_{1.3}$ with $\rho \not\equiv 1$ lead to pairwise similar reduced equations, which take the same form (2.5) if appropriate ansatzes are chosen. Nevertheless, we prefer to use ansatzes 1.3^ρ from Table 2.1 since otherwise Case 1.3 splits into two cases, and without the above explanation, the modified ansatz looks artificial.

The substitution $w_{\tilde z_2\tilde z_2} = h$ maps the modified reduced equation (2.5) to the inviscid Burgers equation

$$h_{\tilde z_1} + hh_{\tilde z_2} = 0,$$

which is the simplest nonlinear transport equation, called also Hopf's equation. An implicit representation of the general solution of this equation is well known, $F(h, \tilde z_2 - h\tilde z_1) = 0$, where F is an arbitrary nonconstant sufficiently smooth function of its arguments. Modulo the G-equivalence, we can assume that w is a fixed second antiderivative of h with respect to $\tilde z_2$. As a result, we construct a family of solutions of (1.1) expressed in terms of quadratures with an implicitly defined function,

- $$u = \int\left(\int h(\tilde z_1, \tilde z_2)\,\mathrm{d}\tilde z_2\right)\mathrm{d}\tilde z_2 - \frac{\rho_t}{6\rho}y^3,$$
$$\tilde z_1 := 2\int\frac{\rho^3-1}{\rho^3}\mathrm{d}t, \quad \tilde z_2 := \frac{y}{\rho} - x, \tag{2.6}$$

where ρ is an arbitrary sufficiently smooth function of t that does not coincide with the constant functions 0 and 1, and the function $h = h(\tilde z_1, \tilde z_2)$

is implicitly defined by the equation $F(h, \tilde z_2 - h\tilde z_1) = 0$ with an arbitrary nonconstant sufficiently smooth function F of its arguments.

Lie and generalized symmetries, cosymmetries, conservation-law characteristics and conservation laws of the inviscid Burgers equation were exhaustively described in Sections 3 and 6 of [123], see also [23, Appendix] for the first computation of the generalized symmetries of this equation. Via the substitution

$$h(\tilde z_1, \tilde z_2) = w_{22}(z_1, z_2) \quad \text{with} \quad \tilde z_1 = 2\int (1-\rho^{-3})\,\mathrm d z_1,\ \ \tilde z_2 = z_2,$$

this results in finding many hidden symmetry-like objects for the equation (1.1).

The normalizer $\mathrm N_{\mathfrak g}(\mathfrak s^\rho_{1.3})$ of the subalgebra $\mathfrak s^\rho_{1.3}$ in $\mathfrak g$ depends on the value of ρ, $\rho_t \neq 0$ and $\rho_t = 0$, respectively,

$$\begin{aligned}
&\mathrm N_{\mathfrak g}(\mathfrak s^\rho_{1.3}) = \big\langle D^{\mathrm s},\, P^x(1),\, P^y(\rho),\, R^y(\beta) - R^x(\rho\beta),\, Z(\sigma)\big\rangle,\\
&\mathrm N_{\mathfrak g}(\mathfrak s^\rho_{1.3}) = \big\langle D^t(1),\, D^t(t),\, D^{\mathrm s},\, P^x(1),\, P^y(\rho),\, R^y(\beta) - R^x(\rho\beta),\, Z(\sigma)\big\rangle,
\end{aligned}$$

where β and σ run through the set of smooth functions of t. The maximal Lie invariance algebra of the modified reduced equation (2.5) is

$$\begin{aligned}
\mathfrak a_{1.3} = \big\langle &\partial_{\tilde z_1},\, \tilde z_1\partial_{\tilde z_1} - w\partial_w,\, \tilde z_1^2\partial_{\tilde z_1} + \tilde z_1\tilde z_2\partial_{\tilde z_2} + \big(\tilde z_1 w + \tfrac16\tilde z_2^3\big)\partial_w,\\
&\tilde z_2\partial_{\tilde z_2} + 3w\partial_w,\, \partial_{\tilde z_2},\, \tilde z_1\partial_{\tilde z_2} + \tfrac12\tilde z_2^2\partial_w,\, \tilde\alpha(\tilde z_1)\tilde z_2\partial_w,\, \tilde\sigma(\tilde z_1)\partial_w\big\rangle,
\end{aligned}$$

where $\tilde\alpha$ and $\tilde\sigma$ run through the set of smooth functions of $\tilde z_1$. The vector fields $D^{\mathrm s}$, $P^x(1)+P^y(\rho)$, $P^y(\rho)$, $R^y(\beta)-R^x(\rho\beta)$, $Z(\sigma)$ and, if $\rho_t=0$, $D^t(1)$ and $D^t(t)$ from $\mathrm N_{\mathfrak g}(\mathfrak s^\rho_{1.3})$ induce the Lie-symmetry vector fields $\tilde z_2\partial_{\tilde z_2}+3w\partial_w$, 0, $\partial_{\tilde z_2}$, $\tilde\alpha\tilde z_2\partial_w$ with $\tilde\alpha(\tilde z_1) = \rho(t)\beta(t)$, $\tilde\sigma\partial_w$ with $\tilde\sigma(\tilde z_1) = \sigma(t)$ and, if $\rho_t = 0$, $\partial_{\tilde z_1}$ and $\tilde z_1\partial_{\tilde z_1} + \frac13\tilde z_2\partial_{\tilde z_2}$ of the modified reduced equation (2.5), respectively. All the elements of $\mathfrak a_{1.3}$ from the set complement of the linear span of the above vector fields from $\mathfrak a_{1.3}$ are genuinely hidden symmetries of the equation (1.1). Note that whether the vector fields $\partial_{\tilde z_1}$ and $\tilde z_1\partial_{\tilde z_1}+\frac13\tilde z_2\partial_{\tilde z_2}$ are genuinely hidden symmetries of (1.1) depends on the value of the parameter function ρ, which does not appear in the modified reduced equation (2.5) and in its maximal Lie invariance algebra $\mathfrak{a}_{1.3}$.

In view of the representation (2.6) for all $\mathfrak{s}^{\rho}_{1.3}$-invariant solutions of the equation (1.1), we do not carry out further Lie reductions of the modified reduced equation (2.5) with respect to subalgebras of $\mathfrak{a}_{1.3}$ although most of them are associated with hidden Lie symmetries of (2.5). At the same time, in Section 2.5.1 we exhaustively study essential direct Lie reductions of (2.5) with respect to two-dimensional subalgebras of $\mathfrak{g}$ that can be interpreted as two-step Lie reductions with reduction 1.3 as the first step.

1.4. $\mathfrak{s}^{\beta}_{1.4} = \big\langle P^x(1) + R^y(\beta)\big\rangle$ with $\beta = \beta(t)$. Each reduced equation 1.4^{β} is trivial, see [92, Section 4.4]. Its general solution is an arbitrary sufficiently smooth function of (z_1, z_2) if $\beta = 0$ and

$$w = \frac{1}{6}\beta_1\beta^{-1}z_2^3 + \varrho^2(z_1)z_2^2 + \varrho^1(z_1)z_2 + \varrho^0(z_1)$$

with arbitrary sufficiently smooth functions ϱ^0, ϱ^1 and ϱ^2 of $z_1 = t$ if $\beta \neq 0$. The case $\beta = 0$ leads to the solution family $u = w(t, y)$ of (1.1), which is parameterized by an arbitrary sufficiently smooth function w of (t, y) and is hence a subfamily of the family (2.1). In the case $\beta \neq 0$, the coefficients ϱ^0, ϱ^1 and ϱ^2 can be assumed, up to the G-equivalence, to vanish. This leads to the following simple solutions of the equation (1.1):

- $$u = \frac{\beta_t}{6\beta}y^3 + \beta xy, \tag{2.7}$$

where β is an arbitrary sufficiently smooth function of t.

We can modify ansatz 1.4^{β} with $\beta \neq 0$ to $u = \tilde{w}(z_1, z_2) + \beta xy + \frac{1}{6}\beta_t\beta^{-1}y^3$ with the same $z_1 = t$ and $z_2 = y$, which simplifies reduced equation 1.4^{β} to $\tilde{w}_{222} = 0$. Therefore, analogously to the subalgebras $\mathfrak{s}^{\rho}_{1.3}$ with $\rho \neq 1$, the subalgebras from the family $\{\mathfrak{s}^{\beta}_{1.4} \mid \beta \neq 0\}$, which is parameterized by an arbitrary nonvanishing function β of t, also correspond, under this ansatz choice, to the same reduced equation.

Depending on the value of the parameter function β, the normalizer $\mathrm{N}_{\mathfrak g}(\mathfrak s^{\beta}_{1.4})$ of the subalgebra $\mathfrak s^{\beta}_{1.4}$ in $\mathfrak g$ is

$$\big\langle D^t(1),\ D^t(t),\ D^{\rm s},\ P^x(1),\ P^y(\rho),\ R^y(\breve\beta),\ Z(\sigma)\big\rangle \quad \text{if} \quad \beta=0,$$

$$\big\langle D^t(1),\ D^t(t)+\tfrac23D^{\rm s},\ P^x(1),\ P^y(\rho)+R^x(\rho\beta),\ R^y(\breve\beta),\ Z(\sigma)\big\rangle$$
$$\text{if} \quad \beta\neq0,\ \beta_t=0,$$

$$\big\langle D^t(1)+\kappa D^{\rm s},\ P^x(1),\ P^y(\rho)+R^x(\rho\beta),\ R^y(\breve\beta),\ Z(\sigma)\big\rangle$$
$$\text{if} \quad \beta_t\neq0,\ \beta_t=\kappa\beta,$$

$$\big\langle D^t(t{+}\mu)+\big(\kappa{+}\tfrac23\big)D^{\rm s},\ P^x(1),\ P^y(\rho)+R^x(\rho\beta),\ R^y(\breve\beta),\ Z(\sigma)\big\rangle$$
$$\text{if} \quad \beta_t\neq0,\ (t{+}\mu)\beta_t=\kappa\beta,$$

$$\big\langle P^x(1),\ P^y(\rho)+R^x(\rho\beta),\ R^y(\breve\beta),\ Z(\sigma)\big\rangle \quad \text{otherwise},$$

where ρ, $\breve\beta$ and σ run through the set of smooth functions of t. In the second, the third and the fourth cases, $\beta=1$, $(\beta,\kappa)=({\rm e}^t,1)$ and $(\beta,\mu)=(|t|^\kappa,0)$ modulo the G-equivalence, respectively.

For any value of the parameter function β, the vector fields $P^x(1)+R^y(\beta)$, $P^y(\rho)+R^x(\rho\beta)$, $R^y(\breve\beta)$ and $Z(\sigma)$ from $\mathrm{N}_{\mathfrak g}(\mathfrak s^{\beta}_{1.4})$ induce the Lie-symmetry vector fields 0, $\rho\partial_{z_2}-\frac12\rho_tz_2^2\partial_w$, $\breve\beta z_2\partial_w$ and $\sigma\partial_w$ of reduced equations 1.4^β, where $\breve\beta$, ρ and σ are arbitrary smooth functions of $z_1=t$. For particular values of β with extension of $\mathrm{N}_{\mathfrak g}(\mathfrak s^{\beta}_{1.4})$, there are the following additional independent inductions:

$$\partial_{z_1},\ z_1\partial_{z_1}+\tfrac13z_2\partial_{z_2},\ z_2\partial_{z_2}+3w\partial_w \quad \text{by} \quad D^t(1),\ D^t(t),\ D^{\rm s} \quad \text{if} \quad \beta=0,$$

$$\partial_{z_1},\ z_1\partial_{z_1}+z_2\partial_{z_2}+2w\partial_w \quad \text{by} \quad D^t(1),\ D^t(t)+\tfrac23D^{\rm s}$$
$$\text{if} \quad \beta\neq0,\ \beta_t=0,$$

$$\partial_{z_1}+\kappa z_2\partial_{z_2}+3\kappa w\partial_w \quad \text{by} \quad D^t(1)+\kappa D^{\rm s} \quad \text{if} \quad \beta_t\neq0,\ \beta_t=\kappa\beta,$$

$$(z_1+\mu)\partial_{z_1}+(\kappa+1)z_2\partial_{z_2}+(3\kappa+2)w\partial_w \quad \text{by} \quad D^t(t+\mu)+(\kappa+\tfrac23)D^{\rm s}$$
$$\text{if} \quad \beta_t\neq0,\ (t+\mu)\beta_t=\kappa\beta,$$

where κ and μ are arbitrary constants with $\kappa\neq0$. If we use the modified ansatz in the case $\beta\neq0$, the expressions for the analogous induced Lie-symmetry vector fields are formally the same up to replacing w by $\tilde{w}$, except the vector field $\rho\partial_{z_2} - \frac{1}{2}\rho_t z_2^2\partial_w$ that should be replaced by $\rho\partial_{z_2} - \frac{1}{2}(\rho_t + \beta_t\beta^{-1}\rho)z_2^2\partial_{\tilde{w}}$.

Since reduced equation 1.4^0 is in fact the identity, it admits any point transformation in the space with coordinates (z_1, z_2, w) as its point symmetry, and any vector field in this space is its Lie-symmetry vector field. The maximal Lie invariance of the modified reduced equation $\tilde{w}_{222} = 0$ for the case $\beta \neq 0$ coincides with the span of the vector fields ∂_{z_1}, ∂_{z_2}, $z_2\partial_{z_2}$, $z_2^2\partial_{z_2} + 2z_2\tilde{w}\partial_{\tilde{w}}$, $\tilde{w}\partial_{\tilde{w}}$, $\partial_{\tilde{w}}$, $z_2\partial_{\tilde{w}}$ and $z_2^2\partial_{\tilde{w}}$ over the (pseudo)ring smooth functions of z_1. Moreover, it is obvious that these reduced equations also possess very wide sets of other symmetry-like objects. Hence for each β, the equation (1.1) admits many hidden symmetry-like objects associated with reduction 1.4^β but they are not of interest in view of the triviality of reduced equations 1.4^β.

2.5. Lie reductions of codimension two

The subalgebras $\mathfrak{s}_{2.1}^{\lambda}$ with $\lambda = -1/3$, $\mathfrak{s}_{2.7}^{\lambda\nu}$, $\mathfrak{s}_{2.8}^{\lambda}$, $\mathfrak{s}_{2.10}^{\beta}$, $\mathfrak{s}_{2.11}$, $\mathfrak{s}_{2.12}$, $\mathfrak{s}_{2.15}^{\delta\nu}$, $\mathfrak{s}_{2.16}^{\delta}$, $\mathfrak{s}_{2.18}^{\rho\beta\sigma}$, $\mathfrak{s}_{2.19}^{\beta^1\beta^2}$, $\mathfrak{s}_{2.20}^{\beta\sigma}$, $\mathfrak{s}_{2.21}^{\alpha\beta^1\beta^2}$, $\mathfrak{s}_{2.22}^{\alpha\beta\sigma}$, $\mathfrak{s}_{2.23}^{\beta\sigma}$, $\mathfrak{s}_{2.24}^{\sigma}$ and $\mathfrak{s}_{2.25}^{\sigma}$ cannot be used for codimension-two Lie reductions of the equation (1.1) since the rank of these subalgebras is less than two.

The subalgebras $\mathfrak{s}_{2.9}^{0}$ and $\mathfrak{s}_{2.17}^{\rho\alpha\beta}$ contain the subalgebras $\mathfrak{s}_{1.4}^{0}$ and $\mathfrak{s}_{1.4}^{\beta}$, respectively, and the one-dimensional subalgebras of $\mathfrak{s}_{2.14}^{\delta 0\delta'}$ and $\mathfrak{s}_{2.6}^{\lambda\delta}$ that are spanned by the respective second basis elements are G-equivalent to $\mathfrak{s}_{1.4}^{\beta}$ for some β. All $\mathfrak{s}_{1.4}^{\beta}$-invariant solutions were constructed in Section 2.4. This is why we can neglect the subalgebras $\mathfrak{s}_{2.6}^{\lambda\delta}$, $\mathfrak{s}_{2.9}^{0}$, $\mathfrak{s}_{2.14}^{\delta 0\delta'}$ and $\mathfrak{s}_{2.17}^{\rho\alpha\beta}$ in the course of carrying out codimension-two Lie reductions of the equation (1.1). The same claim is true for the subalgebras $\mathfrak{s}_{2.5}^{\lambda 1}$, $\mathfrak{s}_{2.9}^{1}$ and $\mathfrak{s}_{2.14}^{\delta 1\delta'}$ due to their relation to the subalgebra $\mathfrak{s}_{1.3}^{1}$.

For the subalgebras $\mathfrak{s}_{2.5}^{\lambda\mu}$, $\mathfrak{s}_{2.9}^{\rho}$ and $\mathfrak{s}_{2.14}^{\delta\nu\delta'}$ with $\mu, \nu \neq 0, 1$ and $\rho \neq 0, 1$, the similar claim is relevant only partially since the representation (2.6) for $\mathfrak{s}^{\rho}_{1.3}$-invariant solutions is not explicit and involves quadratures of the general solution of the inviscid Burgers equation. At the same time, the Lie reductions associated with these subalgebras are simpler and essentially differ from the remaining G-inequivalent Lie reductions. Moreover, we were able to construct general solutions of all the corresponding reduced equations either in an explicit form in terms of elementary functions and the Lambert W function or in a parametric form. This is why we consider the above Lie reductions first.

The remaining subalgebras from the list in Lemma 2.4 are $\mathfrak{s}^{\lambda}_{2.1}$, $\mathfrak{s}^{\nu}_{2.2}$, $\mathfrak{s}^{\nu}_{2.3}$, $\mathfrak{s}_{2.4}$ and $\mathfrak{s}_{2.13}$. They constitute the second collection of subalgebras to be considered within the framework of Lie reductions. All the corresponding invariant solutions are stationary. The integration of the involved reduced equations is more complicated, and the construction of general or even particular solutions in certain closed form is possible only for some of them. For each subalgebra in the second collection, we consider its counterparts among inequivalent two-dimensional subalgebras of the algebra $\mathfrak{g}_{\mathrm{L}}$ and associated G_{L}-inequivalent Lie reductions of the nonlinear Lax representation (1.14). Some of these reductions help us in finding exact solutions to the associated reduced equations.

Below, for each of the subalgebras $\mathfrak{s}^{\lambda\mu}_{2.5}$ with $\mu \neq 0, 1$, $\mathfrak{s}^{\rho}_{2.9}$ with $\rho \neq 0, 1$ and $\mathfrak{s}^{\delta\nu\delta'}_{2.14}$ with $\nu \neq 0, 1$ (the first collection) and $\mathfrak{s}^{\lambda}_{2.1}$ with $\lambda \neq -1/3$, $\mathfrak{s}^{\nu}_{2.2}$, $\mathfrak{s}^{\nu}_{2.3}$, $\mathfrak{s}_{2.4}$ and $\mathfrak{s}_{2.13}$ (the second collection), we present an ansatz for the related invariant solutions and the corresponding reduced equation, where $\varphi = \varphi(\omega)$ is the new unknown functions of the single invariant variable ω. We also compute the subalgebra normalizers and induced symmetries (both infinitesimal and discrete) of reduced equations. In reduced nonlinear Lax representations, $\psi = \psi(\omega)$ is one more unknown functions of the single invariant variable ω, which is associated with the unknown function q in the original nonlinear Lax representation (1.14).

2.5.1. The first collection of reductions. In this subsection, we collect the results for the subalgebras $\mathfrak{s}_{2.5}^{\lambda\mu}$, $\mathfrak{s}_{2.9}$ and $\mathfrak{s}_{2.14}^{\delta\nu\delta'}$.

2.5. $\mathfrak{s}_{2.5}^{\lambda\mu} = \big\langle D^t(1) + \lambda D^{\rm s},\ P^x(\mathrm{e}^{(\lambda-1)t}) + \mu P^y(\mathrm{e}^{(\lambda-1)t})\big\rangle$, $\mu \neq 0, 1$, $|\mu| \leqslant 1$ $(\bmod\ G)$:

$$
\begin{aligned}
&u = \mathrm{e}^{3\lambda t}\varphi - \frac{\lambda-1}{6}(x^3+y^3), \quad \omega = \mathrm{e}^{-\lambda t}(y-\mu x);\\
&2(\mu^3-1)\varphi_{\omega\omega}\varphi_{\omega\omega\omega} - \omega\varphi_{\omega\omega\omega} + (3\lambda-2)\varphi_{\omega\omega} = 0.
\end{aligned}
$$

For any value of the parameter tuple (λ,μ), the subalgebra $\mathfrak{s}_{2.5}^{\lambda\mu}$ has the same normalizer in $\mathfrak{g}$,

$$
\begin{aligned}
\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_{2.5}^{\lambda\mu}) = \big\langle & D^t(1),\ D^{\rm s},\ P^x(\mathrm{e}^{(\lambda-1)t}) + \mu P^y(\mathrm{e}^{(\lambda-1)t}),\\
& -\mu R^x(\mathrm{e}^{2\lambda t}) + R^y(\mathrm{e}^{2\lambda t}),\ Z(\mathrm{e}^{3\lambda t})\big\rangle.
\end{aligned}
$$

Reduced equation $2.5^{\lambda\mu}$ is invariant with respect to the algebra

$$\mathfrak{a}_{2.5} = \langle \omega\partial_\omega + 3\varphi\partial_\varphi, \partial_\varphi, \omega\partial_\varphi\rangle$$

and the point transformation $(\tilde\omega,\tilde\varphi) = (-\omega,-\varphi)$ and, therefore, with respect to the Lie group $G_{2.5}$ that consists of the point transformations $\tilde\omega = a_1\omega$, $\tilde\varphi = a_1^3\varphi + a_2\omega + a_3$, where a_1, a_2 and a_3 are arbitrary constants with $a_1 \neq 0$. The vector fields $D^t(1) + \lambda D^{\rm s}$, $D^{\rm s}$, $P^x(\mathrm{e}^{(\lambda-1)t}) + \mu P^y(\mathrm{e}^{(\lambda-1)t})$, $-\mu R^x(\mathrm{e}^{2\lambda t}) + R^y(\mathrm{e}^{2\lambda t})$ and $Z(\mathrm{e}^{3\lambda t})$ from $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_{2.5}^{\lambda\mu})$ induce the Lie-symmetry vector fields 0, $\omega\partial_\omega + 3\varphi\partial_\varphi$, 0, $\omega\partial_\varphi$ and ∂_φ of reduced equation $2.5^{\lambda\mu}$, respectively. This means that the entire algebra $\mathfrak{a}_{2.5}$ is induced by elements of $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_{2.5}^{\lambda\mu})$. Alternating the signs of (ω,φ) is a discrete point symmetry of reduced equation $2.5^{\lambda\mu}$ and is induced by the discrete point symmetry transformation $\mathcal{J}^{\rm s} := \mathcal{D}^{\rm s}(-1)$ of the original equation (1.1). Hence the group $G_{2.5}$ is entirely induced by the point symmetry group G of the original equation (1.1).

For any values of (λ,μ), reduced equation $2.5^{\lambda\mu}$ is satisfied by all φ with $\varphi_{\omega\omega} = 0$ but such values of φ are $G_{2.5}$-equivalent to 0 and, moreover, corresponds to solutions of the equation (1.1) that are G-equivalent to the zero solution $u = 0$. Further we assume that $\varphi_{\omega\omega} \neq 0$.

There are several values of λ, for which the general solutions of the corresponding reduced equation $2.5^{\lambda\mu}$ can be represented in the closed form. These are $\lambda = 2/3$, $\lambda = 1/3$, $\lambda = 5/6$ and $\lambda = 1$.

Solving of reduced equation $2.5^{\lambda\mu}$ with $\lambda = 2/3$ degenerates to the independent consideration of two equations, the trivial equation $\varphi_{\omega\omega\omega} = 0$ and the equation $2(\mu^3 - 1)\varphi_{\omega\omega} = \omega$. Solutions of the first equation correspond to solutions of the equation (1.1) that are G-equivalent to the zero solution $u = 0$, whereas the second equation is a particular case of the constraint $2(\mu^3 - 1)\varphi_{\omega\omega} = -3(\lambda - 1)\omega$, whose solution set is contained in that of reduced equation $2.5^{\lambda\mu}$ for any λ, and the associated solutions of the equation (1.1) take, modulo the G-equivalence, the form

- $$u = \frac{\kappa}{4(\mu^3 - 1)}(y - \mu x)^3 + \frac{\kappa}{6}(x^3 + y^3) \tag{2.8}$$

with $\kappa := -(\lambda - 1)$. Note that the solution (2.8) with fixed values of κ and μ is in fact invariant with respect to the three-dimensional subalgebra $\big\langle D^t(1),\, D^{\rm s},\, P^x(\mathrm{e}^{-\kappa t}) + \mu P^y(\nu\mathrm{e}^{-\kappa t})\big\rangle$ of $\mathfrak{g}$.

Below $\lambda \neq 2/3$. We can integrate reduced equation $2.5^{\lambda\mu}$ once in two different ways. The first way uses the fact that the left-hand side of this equation is a total derivative with respect to ω, thus leading to the equation

$$(\mu^3 - 1)(\varphi_{\omega\omega})^2 - \omega\varphi_{\omega\omega} + (3\lambda - 1)\varphi_\omega + c_1 = 0, \tag{2.9}$$

where c_1 is the integration constant. The second way is to consider reduced equation $2.5^{\lambda\mu}$ as a first-order ordinary differential equation with respect to $\varphi_{\omega\omega}$, which integrates to

$$\begin{aligned} &\omega = -\frac{2}{3}\frac{\mu^3 - 1}{\lambda - 1}\varphi_{\omega\omega} + c_2|\varphi_{\omega\omega}|^{\frac{1}{3\lambda-2}} \quad \text{if} \quad \lambda \neq 1, \\ &\omega = -2(\mu^3 - 1)\varphi_{\omega\omega}\ln|\varphi_{\omega\omega}| + c_2\varphi_{\omega\omega} \quad \text{if} \quad \lambda = 1, \end{aligned} \tag{2.10}$$

where c_2 is another arbitrary constant, and the integrated equations can be easily solved as algebraic equations with respect to $\varphi_{\omega\omega}$ for four values of λ, $\lambda = 1/3$, $\lambda = 5/6$, $\lambda = 4/3$ and $\lambda = 1$. Some of these values are also singled out in the course of the further integration.

Moreover, the case $\lambda = 1/3$ is singular from the point of view of gauging the constant c_1 by point symmetries of the corresponding reduced equation. To be specific, in contrast to the other values of λ, we cannot set c_1 to be equal to zero and can only assume that $c_1 \in \{-1, 0, 1\}$. The equation (2.9) with $\lambda = 1/3$ is easily solved. Its general solution is

$$\varphi = \frac{\omega^3 + \varepsilon\sqrt{(\omega^2 + c_1)^3}}{12(\mu^3 - 1)} + \frac{\varepsilon c_1}{4(\mu^3 - 1)}\left(\omega \ln\left|\omega + \sqrt{\omega^2 + c_1}\,\right| - \sqrt{\omega^2 + c_1}\right) + c_2\omega + c_3,$$

where $\varepsilon = \pm 1$, and c_2 and c_3 are integration constants, which can be set to be equal zero modulo the $G_{2.5}$-equivalence. The corresponding family of solutions of the equation (1.1) is

- $$u = \mathrm{e}^t \frac{\omega^3 + \varepsilon\sqrt{(\omega^2 + c_1)^3}}{12(\mu^3 - 1)} + \frac{\varepsilon c_1 \mathrm{e}^t}{4(\mu^3 - 1)}\left(\omega \ln\left|\omega + \sqrt{\omega^2 + c_1}\,\right| - \sqrt{\omega^2 + c_1}\right) + \frac{x^3 + y^3}{9},$$

where $\omega = \mathrm{e}^{-t/3}(y - \mu x)$, $\varepsilon = \pm 1$, μ is an arbitrary constant with $\mu \neq 0, 1$, and $c_1 \in \{-1, 0, 1\} \pmod{G_{2.5}}$.

For $\lambda = 5/6$, $\lambda = 4/3$ and $\lambda = 1$, the general solutions of the corresponding reduced equations $2.5^{\lambda\mu}$ can also be represented in closed form, where for convenience we use other integration constants b_1, b_2 and b_3, and $\varepsilon = \pm 1$:

$$\lambda = \frac{5}{6}: \quad \varphi = \frac{\mu^3 - 1}{b_1}\omega^2 + \frac{4\varepsilon}{15 b_1^3}\left(4(\mu^3 - 1)^2 - b_1\omega\right)^{5/2} + b_2\omega + b_3,$$

$$\lambda = \frac{4}{3}: \quad \varphi = -\frac{\omega^3}{12(\mu^3 - 1)} + \frac{b_1^2\omega^2}{16(\mu^3 - 1)^2} + \varepsilon b_1 \frac{\left(b_1^2 - 8(\mu^3 - 1)\omega\right)^{5/2}}{1920(\mu^3 - 1)^4} + b_2\omega + b_3,$$

$$\lambda=1\colon\quad \varphi=-\omega^3\frac{18z^2+15z+4}{216(\mu^3-1)z^3}+b_2\omega+b_3,$$
$$z\in\big\{W_0(\tilde\omega),W_{-1}(\tilde\omega)\big\},\quad \tilde\omega:=-\frac{b_1\omega}{2(\mu^3-1)},$$

where W_0 and W_{-1} are the principal real and the other real branches of the Lambert W function, respectively. The solutions with $b_1=0$ correspond to solutions of the equation (1.1) that belong, up to the G-equivalence, to the family (2.8). Hence we can assume that $b_1\neq 0$ and thus set the gauges $b_1=1,\ b_2=b_3=0\pmod{G_{2.5}}$. This leads to the following G-inequivalent solutions of the equation (1.1) with $\varepsilon=\pm1$ and an arbitrary constant $\mu\neq 0,1$:

- $$u=(\mu^3-1)\mathrm{e}^{\frac56 t}(y-\mu x)^2+\frac{x^3+y^3}{36}$$
$$+\frac{4\varepsilon}{15}\mathrm{e}^{\frac52 t}\big(4(\mu^3-1)^2-\mathrm{e}^{-\frac56 t}(y-\mu x)\big)^{5/2},$$
- $$u=-\frac{(y-\mu x)^3}{12(\mu^3-1)}+\mathrm{e}^{\frac43 t}\frac{(y-\mu x)^2}{16(\mu^3-1)^2}-\frac{x^3+y^3}{18}$$
$$+\varepsilon\mathrm{e}^{4t}\frac{\big(1-8(\mu^3-1)\mathrm{e}^{-\frac43 t}(y-\mu x)\big)^{5/2}}{1920(\mu^3-1)^4},$$
- $$u=-(y-\mu x)^3\frac{18z^2+15z+4}{216(\mu^3-1)z^3},$$
$$z\in\big\{W_0(\tilde\omega),W_{-1}(\tilde\omega)\big\},\quad \tilde\omega:=-\frac{\mathrm{e}^{-t}(y-\mu x)}{2(\mu^3-1)}.$$

For any $\lambda\neq 2/3,1/3,1$, the general solution of reduced equation $2.5^{\lambda\mu}$ can be represented in a parametric form in a uniform way. Considering the derivative $\varphi_{\omega\omega}$ in the equations (2.9) and (2.10) as a parameter and denoting it by s, we rewrite these equations as

$$\omega=-\frac23\frac{\mu^3-1}{\lambda-1}s+c_2|s|^{\frac{1}{3\lambda-2}},\quad \varphi_\omega=-\frac{(\mu^3-1)s^2-\omega s+c_1}{3\lambda-1}.$$

The associated parametric expression for $\varphi=\int\varphi_\omega\,\mathrm{d}\omega$ is given by

$$
\begin{aligned}
\varphi =&(\mu^3-1)\frac{4(\mu^3-1)s-3\omega}{27\lambda(2\lambda-1)}s^2-\frac{(\mu^3-1)s-\omega}{3\lambda(3\lambda-1)}\omega s\\
&-\frac{c_1\omega}{3\lambda-1}+c_3\quad\text{if}\quad\lambda\neq 0,\frac12,\frac13,\\
\varphi =&\frac{2}{27}(\mu^3-1)^2s^3-\frac59 c_2(\mu^3-1)|s|^{3/2}+\frac{c_2^2}{2}\operatorname{sgn}(s)\ln|s|\\
&+c_1\omega+c_3\quad\text{if}\quad\lambda=0,\\
\varphi =&\frac{8}{27}(\mu^3-1)^2s^3+\frac43 c_2(\mu^3-1)\ln|s|+\frac{4c_2^2}{3s^3}\\
&+2c_1\omega+c_3\quad\text{if}\quad\lambda=\frac12.
\end{aligned}
\tag{2.11}
$$

Note that the value $c_2=0$ corresponds to solutions of the form (2.8) and can be excluded from the consideration. Thus, we can assume $c_2\neq 0$ and thus set $c_2=1$, $c_1=c_3=0$ (mod $G_{2.5}$). This leads to the following G-inequivalent solutions of the equation (1.1):

- $u=\mathrm{e}^{3\lambda t}\varphi-\dfrac{\lambda-1}{6}(x^3+y^3),$

where $\lambda\neq 2/3,1/3,1$, $\mu\neq 0,1$, and the function φ is defined by the appropriate equation from (2.11) with $c_2=1$, $c_1=c_3=0$, $\omega:=\mathrm{e}^{-\lambda t}(y-\mu x)$, and additionally the function $s=s(\omega)$ is implicitly defined as a solution of the Lambert's transcendental equation

$$
|s|^{\frac{1}{3\lambda-2}}-\frac23\frac{\mu^3-1}{\lambda-1}s=\omega. \tag{2.12}
$$

In fact, the elementary solvability of this equation for $\lambda\in\{5/6,4/3\}$ as a quadratic equation with respect to a degree of s has been used above for deriving explicit solutions of the equation (1.1). The equation (2.12) can also be solved for some other values of λ as algebraic equations with respect to certain degrees of s, which results in explicit expressions for the general solutions of the corresponding reduced equation $2.5^{\lambda\mu}$. Thus, $(3\lambda-2)^{-1}=-1/2,-2,3,1/3$ for $\lambda=0,1/2,7/9,5/3$, respectively, and the corresponding equations (2.12) are cubic equations with respect to certain degrees of s and, therefore, can be solved, e.g., using the Cardano formula.

2.9. $\mathfrak{s}^{\rho}_{2.9}=\langle D^{\rm s},\ P^x(1)+P^y(\rho)\rangle$, $\rho\not\equiv 1$ for any open interval in the domain of ρ and $\rho(t)\neq 0$ for any t in this domain:

$$u=\frac{(y-\rho x)^3}{\rho^3}\varphi-\frac{\rho_t}{6\rho}y^3,\quad \omega=t;\qquad \varphi_\omega=-12\frac{\rho^3-1}{\rho^3}\varphi^2.$$

The normalizer of the subalgebra $\mathfrak{s}^{\rho}_{2.9}$ in $\mathfrak{g}$ is

$$\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}^{\rho}_{2.9})=\langle D^{\rm s},\ P^x(1)+P^y(\rho)\rangle\quad\text{if}\quad \rho_t\neq 0,$$
$$\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}^{\rho}_{2.9})=\langle D^t(1),\ D^t(t),\ D^{\rm s},\ P^x(1)+P^y(\rho)\rangle\quad\text{if}\quad \rho_t=0.$$

Since reduced equation 2.9^ρ is a first-order ordinary differential equation, its maximal Lie invariance algebra $\mathfrak{a}^{\rho}_{2.9}$ is infinite-dimensional. The normalizer $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}^{\rho}_{2.9})$ induces merely the zero subalgebra of the algebra $\mathfrak{a}^{\rho}_{2.9}$ and its subalgebra $\langle\partial_\omega,\omega\partial_\omega-\varphi\partial_\varphi\rangle$ if $\rho_t\neq 0$ and $\rho_t=0$, respectively. Therefore, the original equation (1.1) admits an infinite number of linearly independent hidden symmetries that are associated with reduced equation 2.9^ρ. Nevertheless, these hidden symmetries are not of great interest in view of the trivial integrability of reduced equation 2.9^ρ. Separating the variables in the reduced equation, we integrate it and substitute the obtained expression for φ into the ansatz, which gives the following a family of solutions of (1.1) (cf. reduction 1.3):

- $u=\left(\displaystyle\int\frac{\rho^3-1}{\rho^3}{\rm d}t\right)^{-1}\dfrac{(y-\rho x)^3}{12\rho^3}-\dfrac{\rho_t}{6\rho}y^3.$

2.14. $\mathfrak{s}^{\delta\nu\delta'}_{2.14}=\langle D^t(1)+\delta D^{\rm s},\ P^x({\rm e}^{\delta t})+\nu P^y({\rm e}^{\delta t})+\delta' R^y({\rm e}^{2\delta t})\rangle$, $\delta,\delta'\in\{0,1\}$, $\nu\neq 0,1$, $|\nu|\leqslant 1$ $(\mathrm{mod}\ G)$:

$$u={\rm e}^{3\delta t}\varphi-\frac{\delta}{6}(x^3+y^3)+\frac{\delta'}{2\nu}{\rm e}^{\delta t}y^2,\quad \omega={\rm e}^{-\delta t}(y-\nu x);$$
$$2\nu(\nu^3-1)\varphi_{\omega\omega}\varphi_{\omega\omega\omega}=\delta'\varphi_{\omega\omega\omega}-3\nu\delta\varphi_{\omega\omega}.$$

Depending on values of (δ,δ'), the normalizer of the subalgebra $\mathfrak{s}_{2.14}^{\delta\nu\delta'}$ in $\mathfrak{g}$ is one of the following:

$$
\begin{aligned}
\mathrm{N}_\mathfrak{g}(\mathfrak{s}_{2.14}^{0\nu0}) &= \langle D^t(1),\, D^t(t),\, D^\mathrm{s},\, P^x(1),\, P^y(1),\, -\nu R^x(1)+R^y(1),\, Z(1)\rangle,\\
\mathrm{N}_\mathfrak{g}(\mathfrak{s}_{2.14}^{0\nu1}) &= \langle D^t(1),\, D^t(t)+\tfrac{2}{3}D^\mathrm{s},\, P^x(1),\, P^y(1)+R^x(1),\\
&\qquad -\nu R^x(1)+R^y(1),\, Z(1)\rangle,\\
\mathrm{N}_\mathfrak{g}(\mathfrak{s}_{2.14}^{1\nu0}) &= \langle D^t(1),\, D^\mathrm{s},\, P^x(\mathrm{e}^t),\, P^y(\mathrm{e}^t),\, -\nu R^x(\mathrm{e}^{2t})+R^y(\mathrm{e}^{2t}),\, Z(\mathrm{e}^{3t})\rangle,\\
\mathrm{N}_\mathfrak{g}(\mathfrak{s}_{2.14}^{1\nu1}) &= \langle D^t(1)+D^\mathrm{s},\, P^x(\mathrm{e}^t),\, P^y(\mathrm{e}^t)+R^x(\mathrm{e}^{2t}),\\
&\qquad -\nu R^x(\mathrm{e}^{2t})+R^y(\mathrm{e}^{2t}),\, Z(\mathrm{e}^{3t})\rangle.
\end{aligned}
$$

Therefore, the vector fields $D^t(1)+\delta D^\mathrm{s}$, $P^x(\mathrm{e}^{\delta t})$, $P^y(\mathrm{e}^{\delta t})+\delta' R^x(\mathrm{e}^{2\delta t})$, $-\nu R^x(\mathrm{e}^{2\delta t})+R^y(\mathrm{e}^{2\delta t})$ and $Z(\mathrm{e}^{3\delta t})$ belong to $\mathrm{N}_\mathfrak{g}(\mathfrak{s}_{2.14}^{\delta\nu\delta'})$ with the corresponding value of (δ,δ') and induce the Lie-symmetry vector fields 0, $-\nu\partial_\omega$, $\partial_\omega-\delta'\nu\omega\partial_\varphi$, $\omega\partial_\varphi$ and ∂_φ of reduced equation $2.14^{\delta\nu\delta'}$, respectively. For any values of (δ,ν,δ'), reduced equation $2.14^{\delta\nu\delta'}$ is invariant with respect to the algebra $\mathfrak{a}_{2.14}=\langle\partial_\omega,\partial_\varphi,\omega\partial_\varphi\rangle$ and, therefore, with respect to the corresponding Lie group $G_{2.14}$, which consists of the point transformations $\tilde\omega=\omega+a_1$, $\tilde\varphi=\varphi+a_2\omega+a_3$, where a_1, a_2 and a_3 are arbitrary constants. The group $G_{2.14}$ is entirely induced by the point symmetry group G of the original equation (1.1). For any values of (δ,ν,δ'), reduced equation $2.14^{\delta\nu\delta'}$ is satisfied by all φ with $\varphi_{\omega\omega}=0$ but such values of φ are $G_{2.14}$-equivalent to 0 and, moreover, correspond to solutions of the equation (1.1) that are G-equivalent to the zero solution $u=0$. Further we can consider only solutions with $\varphi_{\omega\omega}\neq0$.

Consider the case $\delta=0$. Reduced equation $2.14^{0\nu\delta'}$ degenerates to the elementary equation $\varphi_{\omega\omega\omega}=0$ whose maximal Lie invariance algebra is well known, $\mathfrak{a}_{2.14}^{0\nu\delta'}=\langle\partial_\omega,\,\omega\partial_\omega,\,\omega^2\partial_\omega+2\omega\varphi\partial_\varphi,\,\partial_\varphi,\,\omega\partial_\varphi,\,\omega^2\partial_\varphi,\,\varphi\partial_\varphi\rangle$. In addition to the Lie-symmetry vector fields ∂_ω, ∂_φ and $\omega\partial_\varphi$, which are induced as in the general case, elements of the normalizer $\mathrm{N}_\mathfrak{g}(\mathfrak{s}_{2.14}^{0\nu\delta'})$ induce $\omega\partial_\omega+2\varphi\partial_\varphi$ if $\delta'=1$ and $\omega\partial_\omega$ and $\varphi\partial_\varphi$ if $\delta'=0$. Any element of $\mathfrak{a}_{2.14}^{0\nu\delta'}$ involving at least one of the basis vector fields $\omega^2\partial_\omega+2\omega\varphi\partial_\varphi$, $\omega^2\partial_\varphi$ and, if $\delta'=1$,

$\omega\partial_\omega + a\varphi\partial_\varphi$ with $a \neq 2$ is a hidden symmetry of the equation (1.1). All the corresponding solutions of the equation (1.1) are G-equivalent to either the zero solution $u = 0$ or solutions of the form (2.7) with $\beta = \text{const}$.

Reduced equation $2.14^{1\nu0}$ is factored out to $\big(2(\nu^3-1)\varphi_{\omega\omega\omega}+3\big)\varphi_{\omega\omega} = 0$. Therefore, its solution set is the disjoint union of the solution sets of the equations $2(\nu^3-1)\varphi_{\omega\omega\omega} + 3 = 0$ and $\varphi_{\omega\omega} = 0$. This implies that the maximal Lie invariance algebra $\mathfrak{a}^{1\nu0}_{2.14}$ of reduced equation $2.14^{1\nu0}$ is the intersection of the maximal Lie invariance algebras of the above equations, $\mathfrak{a}^{1\nu0}_{2.14} = \langle\partial_\omega,\, \partial_\varphi,\, \omega\partial_\varphi,\, \omega\partial_\omega + 3\varphi\partial_\varphi\rangle$. The entire algebra $\mathfrak{a}^{1\nu0}_{2.14}$ is induced by $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}^{1\nu0}_{2.14})$. Thus, the case $(\delta,\delta') = (1,0)$ leads, modulo the G-equivalence, to the solutions of the equation (1.1) that are of the form (2.8) with $\kappa = -1$ and $\mu = \nu$.

Consider the case $\delta\delta' \neq 0$. Thus, $\delta = 1$. We neglect the gauge $\delta' = 1$, set δ' to another value, $\delta' = -\nu(\nu^3-1)$, and denote $\kappa := -3(\nu^3-1)^{-1}$. As a result, we need to solve the equation

$$\big((\varphi_{\omega\omega})^2 + \varphi_{\omega\omega} - \kappa\varphi_\omega\big)_\omega = 0.$$

We integrate it once, deriving $(\varphi_{\omega\omega})^2 + \varphi_{\omega\omega} - \kappa\varphi_\omega - c_1 = 0$, c_1 is an integration constant, and solve the integrated equation with respect to $\varphi_{\omega\omega}$,

$$\varphi_{\omega\omega} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{4\kappa\varphi_\omega + 1 + 4c_1}.$$

The maximal Lie invariance algebra of reduced equation $2.14^{1\nu\delta'}$ coincides with the common invariance algebra $\mathfrak{a}_{2.14}$ of case 2.14. Modulo the induced $G_{2.14}$-equivalence, we can set $1 + 4c_1 = 0$. Separating the variables in the resulting equation, denoting $z := -1 \pm \sqrt{4\kappa\varphi_\omega}$ and integrating once more, we obtain $z + \ln|z| = \kappa(\omega + c_2)$, where c_2 is another integration constant that also can be set to be equal zero up to the $G_{2.14}$-equivalence. In other words, we derive the equation

$$\omega = F(\varphi_\omega) := \left.\frac{z + \ln|z|}{\kappa}\right|_{z=-1\pm\sqrt{4\kappa\varphi_\omega}},$$

and its general solution can be represented in parametric form as

$$\omega = F(\zeta) \quad \text{with} \quad \zeta := \frac{(z+1)^2}{4\kappa},$$

$$\varphi = \int \zeta \frac{\mathrm{d}F}{\mathrm{d}\zeta}(\zeta)\,\mathrm{d}\zeta = \int \frac{(z+1)^3}{4\kappa^2 z}\,\mathrm{d}z = \frac{1}{4\kappa^2}\left(\frac{z^3}{3} + \frac{3}{2}z^2 + 2z\right) + \frac{\omega}{4\kappa} + c_3,$$

where c_3 is one more integration constant that also can be set to be equal to zero up to the $G_{2.14}$-equivalence. The summand $\omega/(4\kappa)$ can be neglected using the $G_{2.14}$-equivalence as well. Since $z\mathrm{e}^z = \pm\mathrm{e}^{\kappa\omega}$, we have that $z \in \{W_0(\mathrm{e}^{\kappa\omega}),\, W_0(-\mathrm{e}^{\kappa\omega}),\, W_{-1}(-\mathrm{e}^{\kappa\omega})\}$, where W_0 and W_{-1} again denote the principal real and the other real branches of the Lambert W function, respectively. As a result, we show that any solution of reduced equation $2.14^{\delta\nu\delta'}$ with $\delta\delta' \neq 0$ is $G_{2.14}$-equivalent to one of the solutions

$$\varphi = \frac{1}{4\kappa^2}\left(\frac{z^3}{3} + \frac{3}{2}z^2 + 2z\right), \qquad z \in \{W_0(\mathrm{e}^{\kappa\omega}),\, W_0(-\mathrm{e}^{\kappa\omega}),\, W_{-1}(-\mathrm{e}^{\kappa\omega})\}.$$

The corresponding G-equivalent solutions of the equation (1.1) take the form

- $u = \dfrac{(\nu^3-1)^2}{36}\mathrm{e}^{3t}\left(\dfrac{z^3}{3} + \dfrac{3}{2}z^2 + 2z\right) - \dfrac{1}{6}(x^3+y^3) - \dfrac{\nu^3-1}{2}\mathrm{e}^t y^2,$

where $z \in \{W_0(\mathrm{e}^{\kappa\omega}),\, W_0(-\mathrm{e}^{\kappa\omega}),\, W_{-1}(-\mathrm{e}^{\kappa\omega})\}$ with $\omega := \mathrm{e}^{-t}(y-\nu x)$ and $\kappa := -3(\nu^3-1)^{-1}$.

2.5.2. The second collection of reductions. The reduced ordinary differential equations that are obtained from the equation (1.1) by Lie reductions using the two-dimensional subalgebras from the second selected collection are cumbersome and among them there are three one-parameter families of equations, which complicates the computation of Lie and, moreover, point symmetries of these equations. At the same time, there is a more essential obstacle even for computing Lie symmetries just using the standard Lie approach augmented with specialized computer-algebra packages. The general form of the above reduced equations is

$$M(\omega, \varphi, \varphi_\omega, \varphi_{\omega\omega})\varphi_{\omega\omega\omega} + N(\omega, \varphi, \varphi_\omega, \varphi_{\omega\omega}) = 0,$$

where M and N are respectively specific first- and second-degree polynomials in $(\varphi,\varphi_\omega,\varphi_{\omega\omega})$ with coefficients polynomially depending on ω that in addition satisfy the conditions

$$\begin{aligned}&M_{\varphi_{\omega\omega}}\neq 0\quad\text{or}\\&M_{\varphi_{\omega\omega}}=0,\, M_{\varphi_\omega}(3M_{\varphi_\omega}+N_{\varphi_{\omega\omega}\varphi_{\omega\omega}})(6M_{\varphi_\omega}+N_{\varphi_{\omega\omega}\varphi_{\omega\omega}})\neq 0.\end{aligned}\tag{2.13}$$

Some of these equations cannot be represented in normal form due to their degeneration, and the solution set of each of them splits into two parts that are singled out by the constraints $M\neq 0$ and $M=N=0$, respectively. The left-hand sides of several of them even admit algebraic factorizations. Therefore, the maximal Lie invariance algebra of such an equation $M\varphi_{\omega\omega\omega}+N=0$ is the intersection of the maximal Lie invariance algebras of the equation $\varphi_{\omega\omega\omega}=-N/M$ with $M\neq 0$ and of the (overdetermined) system $M=N=0$. We prove that under the conditions (2.13), any Lie-symmetry vector field of the equation $\varphi_{\omega\omega\omega}=-N/M$ is necessarily of the form $\xi\partial_\omega+(\eta^1\varphi+\eta^0)\partial_\varphi$, where ξ, η^1 and η^0 are functions of ω, and then the computation of the maximal Lie invariance algebra of this equation can be easily completed with a computer-algebra system even in the case of presence of a parameter. After reducing the corresponding system $M=N=0$ to a passive form, we also find its maximal Lie invariance algebra if its solution set is nonempty. For each family of reduced equations under study in this section, the construction of its point symmetry group is specific and is carried out using the algebraic method by Hydon [60–62].

2.1. $\mathfrak s^\lambda_{2.1}=\big\langle D^t(1),\, D^t(t)+\lambda D^{\rm s}\big\rangle$, $\lambda\neq -1/3$:

$$u=|x|^\kappa\varphi,\quad \omega=\frac{y}{x},\quad \kappa:=\frac{9\lambda}{3\lambda+1};$$

$$\begin{aligned}&\big(2\omega(\omega^3-1)\varphi_{\omega\omega}-(\kappa-1)(3\omega^3-1)\varphi_\omega+\kappa(\kappa-1)\omega^2\varphi\big)\varphi_{\omega\omega\omega}\\&\quad-(\kappa-2)\big((5\omega^3-1){\varphi_{\omega\omega}}^2-(\kappa-1)\omega(11\omega\varphi_\omega-3\kappa\varphi)\varphi_{\omega\omega}\\&\quad+(\kappa-1)^2(5\omega\varphi_\omega-2\kappa\varphi)\varphi_\omega\big)=0.\end{aligned}$$

In view of its definition, the parameter κ cannot be equal to 3 but we can neglect this fact by uniting reduction 2.1 with reduction 2.13, which can be considered as corresponding to the values $\lambda = \pm\infty$ and $\kappa = 3$, see below. Note that it is convenient to assume the family of reduced equations 2.1 to be parameterized κ instead of λ.

The associated system $M = N = 0$ is equivalent to the equation

$$\begin{aligned}
&\varphi_\omega = 0 \quad \text{if} \quad \kappa = 0,\\
&\varphi_{\omega\omega} = 0 \quad \text{if} \quad \kappa = 1,\\
&2\omega(\omega^3 - 1)\varphi_{\omega\omega} - (3\omega^3 - 1)\varphi_\omega + 2\omega^2\varphi = 0 \quad \text{if} \quad \kappa = 2,\\
&(w^3 + 1)\varphi_\omega = 3\omega^2\varphi \quad \text{if} \quad \kappa = 3,\\
&(w^6 - 10w^3 + 1)\varphi_\omega = 6\omega^2(w^3 - 5)\varphi \quad \text{if} \quad \kappa = 6,\\
&\varphi = 0 \quad \text{otherwise.}
\end{aligned}$$

The corresponding solutions of the original equation (1.1) belong to the family of trivial solutions (2.1), except the cases $\kappa = 2$, see the consideration of this case below, and $\kappa = 6$, with the polynomial solutions

$$u = c_1(x^6 - 10x^3y^3 + y^6)$$

of (1.1), where c_1 is an arbitrary constant.

The normalizer of the subalgebra $\mathfrak{s}_{2.1}^\lambda$ in $\mathfrak{g}$ is

$$\begin{aligned}
&\mathrm{N}_\mathfrak{g}(\mathfrak{s}_{2.1}^\lambda) = \langle D^t(1), D^t(t), D^\mathrm{s}\rangle \quad \text{if} \quad \lambda \neq 0, 1/6,\\
&\mathrm{N}_\mathfrak{g}(\mathfrak{s}_{2.1}^0) = \langle D^t(1), D^t(t), D^\mathrm{s}, Z(1)\rangle,\\
&\mathrm{N}_\mathfrak{g}(\mathfrak{s}_{2.1}^{1/6}) = \langle D^t(1), D^t(t), D^\mathrm{s}, R^x(1), R^y(1)\rangle.
\end{aligned}$$

For a general value $\lambda \neq -1/3$, the vector fields $D^t(1)$, $3D^t(t)$ and D^s induce the Lie-symmetry vector fields 0, $-\kappa\varphi\partial_\varphi$ and $(3-\kappa)\varphi\partial_\varphi$ of reduced equation 2.1^κ, whereas for $\lambda = 0$ and $\lambda = 1/6$ (i.e., $\kappa = 0$ and $\kappa = 1$) we in addition have inductions ∂_φ by $Z(1)$ and ∂_φ and $\omega\partial_\varphi$ by $R^x(1)$ and $R^y(1)$, respectively. The discrete point symmetry transformations $\mathcal{J}$ and $\mathcal{J}^\mathrm{s}$

of the equation (1.1), see Corollary 1.4, induce the discrete point symmetry transformations

$$(\tilde\omega,\tilde\varphi)=(\omega^{-1},|\omega|^{-\kappa}\varphi)\quad\text{and}\quad(\tilde\omega,\tilde\varphi)=(\omega,-\varphi)$$

of reduced equation 2.1^κ for any κ, respectively, whereas the discrete point symmetry transformation $\mathcal{J}^{\rm i}$ of the equation (1.1) corresponds to the identity transformation of (ω,φ).

For a general value $\lambda\neq-1/3$, the subalgebra $\mathfrak{s}^\lambda_{2.1}$ has, up to the $G_{\rm L}$-equivalence, a single counterpart among subalgebras of the algebra $\mathfrak{g}_{\rm L}$, $\bar{\mathfrak{s}}^\lambda_{2.1}=\big\langle\bar D^t(1),\,\bar D^t(t)+\lambda\bar D^{\rm s}\big\rangle$. This is why ansatz 2.1 is extended to v as $v=|x|^{\kappa/2}\psi$, and the nonlinear Lax representation (1.14) reduces to the system

$$\begin{aligned}&12(\kappa-1)\big(2(\omega^3+1)\psi_\omega-\kappa\omega^2\psi\big)\varphi_\omega\\&\qquad-12\kappa(\kappa-1)\omega\big(2\omega\psi_\omega-\kappa\psi\big)\varphi+16\omega(\omega^3-1)\psi_\omega^{\,3}\\&\qquad-12\kappa(\omega^3-1)\psi\psi_\omega^{\,2}+\kappa^3\omega\psi^3=0,\\&\omega\varphi_{\omega\omega}-(\kappa-1)\varphi_\omega+\omega\psi_\omega^{\,2}-\frac{\kappa}{2}\psi\psi_\omega=0.\end{aligned}\tag{2.14}$$

For each of the values $\lambda=0$ and $\lambda=2/3$, where $\kappa=0$ and $\kappa=2$, there is another counterpart of the subalgebra $\mathfrak{s}^\lambda_{2.1}$ among subalgebras of the algebra $\mathfrak{g}_{\rm L}$ that is $G_{\rm L}$-inequivalent to the subalgebra $\bar{\mathfrak{s}}^\lambda_{2.1}$,

$$\bar{\mathfrak{s}}^0_{2.1'}=\big\langle\bar D^t(1),\,\bar D^t(t)+\frac13\bar P^v\big\rangle\quad\text{and}\quad\bar{\mathfrak{s}}^{2/3}_{2.1'}=\big\langle\bar D^t(1)+\bar P^v,\,\bar D^t(t)+\frac23\bar D^{\rm s}\big\rangle,$$

respectively. This results in one more $G_{\rm L}$-inequivalent extension of ansatz 2.1 to v for each of these values of λ, $v=\psi+\ln|x|$ and $v=x\psi+t$. The corresponding reduced systems are

$$\begin{aligned}&3\big((\omega^3+1)\psi_\omega-\omega^2\big)\varphi_\omega-2\omega(\omega^3-1)\psi_\omega^{\,3}+3(\omega^3-1)\psi_\omega^{\,2}-\omega=0,\\&\omega\varphi_{\omega\omega}+\varphi_\omega+\omega\psi_\omega^{\,2}-\psi_\omega=0\end{aligned}$$

and

$$\begin{aligned}&3\big((\omega^3+1)\psi_\omega-\omega^2\psi\big)\varphi_\omega-6\omega(\omega\psi_\omega-\psi)\varphi\\&\qquad+2\omega(\omega^3-1)\psi_\omega^{\,3}-3(\omega^3-1)\psi\psi_\omega^{\,2}+\omega\psi^3-3\omega=0,\\&\omega\varphi_{\omega\omega}-\varphi_\omega+\omega\psi_\omega^{\,2}-\psi\psi_\omega=0.\end{aligned}$$

For the specific values $\kappa \in \{0,1,2\}$ or, equivalently, $\lambda \in \{0,1/6,2/3\}$, we are able to construct more solutions than for the other values.

$\kappa=2$. The solution set of reduced equation 2.1^κ with $\kappa=2$, i.e., $\lambda=2/3$, is a union of the solution sets of the equations

$$\varphi_{\omega\omega\omega}=0 \quad \text{and} \quad 2\omega(\omega^3-1)\varphi_{\omega\omega}-(3\omega^3-1)\varphi_\omega+2\omega^2\varphi=0,$$

whose intersection consists only of the zero solution $\varphi=0$. It can be proved that the maximal Lie invariance algebra $\mathfrak{a}^2_{2.1}$ of reduced equation 2.1^2 is the intersection of the maximal Lie invariance algebras of the above equations, $\mathfrak{a}^2_{2.1}=\langle\varphi\partial_\varphi\rangle$, and thus it is induced by $\mathrm{N}_\mathfrak{g}(\mathfrak{s}^2_{2.1})$.

The solutions of the first equation are not interesting since each related solution of the equation (1.1) is G-equivalent to either the zero solution $u=0$ or the solution (2.7) with $\beta=1$. The general solution of the second equation is

$$\begin{aligned}&\varphi=c_1(\omega^{3/2}+1)^{4/3}+c_2(\omega^{3/2}-1)^{4/3} \quad \text{for} \quad \omega\geqslant 0,\\ &\varphi=(1-\omega^3)^{2/3}\left(c_1\cos\left(\frac43\arctan|\omega|^{3/2}\right)+c_2\sin\left(\frac43\arctan|\omega|^{3/2}\right)\right)\\ &\qquad \text{for} \quad \omega\leqslant 0,\end{aligned}$$

where c_1 and c_2 are arbitrary constants, and one of them, if nonzero, can be set to one by induced symmetries of reduced equation 2.1^2. This leads to the following solutions of the equation (1.1):

- $u=c_1(|x|^{3/2}+|y|^{3/2})^{4/3}+c_2(|y|^{3/2}-|x|^{3/2})^{4/3} \quad \text{for} \quad xy\geqslant 0,$
- $u=(x^3-y^3)^{2/3}\left(c_1\cos\left(\frac43\arctan\left|\frac{y}{x}\right|^{3/2}\right)+c_2\sin\left(\frac43\arctan\left|\frac{y}{x}\right|^{3/2}\right)\right)$
  for $xy\leqslant 0,$

where c_1 and c_2 are arbitrary constants, and one of them, if nonzero, can be set to one up to the G-equivalence.

$\kappa=1$. The solution set of reduced equation 2.1^κ with $\kappa=1$, i.e., $\lambda=1/6$, coincides with that of the equation

$$2\omega(\omega^3-1)\varphi_{\omega\omega\omega}+(5\omega^3-1)\varphi_{\omega\omega}=0$$

and thus consists of the functions

$$\varphi=c_1\varphi^0(\omega)+c_2\omega+c_3\quad\text{with}\quad\varphi^0(\omega):=|\omega|^{3/2}\,{}_3F_2\left(\frac16,\frac12,\frac23;\frac76,\frac32;w^3\right),$$

where ${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)$ is the generalized hypergeometric function. Hence the maximal Lie invariance algebra of reduced equation 2.1^1 is $\mathfrak{a}^1_{2.1}=\langle\partial_\varphi,\omega\partial_\varphi,\varphi^0(\omega)\partial_\varphi,\varphi\partial_\varphi\rangle$, and its subalgebra induced by $\mathrm{N}_\mathfrak{g}(\mathfrak{s}^1_{2.1})$ is $\langle\partial_\varphi,\omega\partial_\varphi,\varphi\partial_\varphi\rangle$, i.e., any element of $\mathfrak{a}^1_{2.1}$ with nonzero coefficient of $\varphi^0(\omega)\partial_\varphi$ is a hidden Lie-symmetry of the equation (1.1) that is associated with reduction 2.1^1.

Up to induced symmetries of reduced equation 2.1^1, we can set $c_1=1$, and $c_2=c_3=0$. Thus, the only corresponding G-inequivalent solutions of the equation (1.1) is

- $$u=\sqrt{\left|\frac{y^3}{x}\right|}\,{}_3F_2\left(\frac16,\frac12,\frac23;\frac76,\frac32;\frac{y^3}{x^3}\right).$$

$\kappa=0$. The maximal Lie invariance algebra $\mathfrak{a}^0_{2.1}$ of reduced equation 2.1^0 is equal to $\langle\partial_\varphi,\varphi\partial_\varphi\rangle$, and thus it is entirely induced by $\mathrm{N}_\mathfrak{g}(\mathfrak{s}^0_{2.1})$. The reduced system (2.14) with $\kappa=0$ degenerates to

$$\begin{aligned}&\psi_\omega\big(3(\omega^3+1)\varphi_\omega-2\omega(\omega^3-1){\psi_\omega}^2\big)=0,\\&\omega\varphi_{\omega\omega}+\varphi_\omega+\omega{\psi_\omega}^2=0.\end{aligned}\tag{2.15}$$

It is obvious that the integration of the system (2.15) splits into two cases. If $\psi_\omega=0$, then it is equivalent to the equation $\omega\varphi_{\omega\omega}+\varphi_\omega=0$, whose general solution is $\varphi=c_1\ln|\omega|+c_0$ and gives only trivial solutions of (1.1) from the family (2.1). Under the constraint $\psi_\omega\neq0$, we easily exclude ψ_ω from the system (2.15) and derive the equation $2\omega(\omega^3-1)\varphi_{\omega\omega}+(5\omega^3+1)\varphi_\omega=0$.

The general solution of this equation is

$$\varphi = c_1 \ln \left| \frac{\omega^{3/2} + 1}{\omega^{3/2} - 1} \right| + c_2 \quad \text{for} \quad \omega \geqslant 0,$$

$$\varphi = c_1 \arctan |\omega|^{3/2} + c_2 \quad \text{for} \quad \omega \leqslant 0.$$

Up to induced symmetries of reduced equation 2.1^0, we can set $c_1 = 1$, and $c_2 = 0$. This leads to the following solutions of the equation (1.1):

- $u = \ln \left| \dfrac{|x|^{3/2} + |y|^{3/2}}{|x|^{3/2} - |y|^{3/2}} \right| \quad \text{for} \quad xy \geqslant 0,$
- $u = \arctan \left| \dfrac{y}{x} \right|^{3/2} \quad \text{for} \quad xy \leqslant 0.$

The independent variable ω and the ratio $\varphi_{\omega\omega}/\varphi_\omega$ are the lowest-order differential invariants of the solvable algebra $\mathfrak{a}^0_{2.1}$. Therefore, the change of dependent variable $p = \varphi_{\omega\omega}/\varphi_\omega$ lowers the order of reduced equation 2.1^0 by two. The derived equation

$$(2\omega(\omega^3-1)p+3\omega^3-1)p_\omega+2\omega(\omega^3-1)p^3+(13\omega^3-3)p^2+22p\omega^2+10\omega=0$$

integrates to

$$\frac{\big(\omega(\omega^3-1)^2p^2+(3\omega^3-1)(\omega^3-1)p+\omega^2(2\omega^3-5)\big)^3}{\big(2\omega(\omega^3-1)p+5\omega^3+1\big)^4\big(\omega p+1\big)^2} = c_1.$$

Substituting $\varphi_{\omega\omega}/\varphi_\varphi$ for p into the last equation, we obtain the first integral of reduced equation 2.1^0. We are not able to integrate further for the general value of c_1. Nevertheless, setting $c_1 = 0$ simplifies the equation into be integrated to the equation

$$\omega(\omega^3-1)^2\varphi_{\omega\omega}^{\;2}+(3\omega^3-1)(\omega^3-1)\varphi_{\omega\omega}\varphi_\omega+\omega^2(2\omega^3-5)\varphi_\omega^{\;2}=0,$$

which is easily solved as a quadratic equation with respect to $\varphi_{\omega\omega}/\varphi_\omega$ and integrated twice. As a result, we construct the following solution of reduced equation 2.1^κ with $\kappa = 0$:

$$\varphi = \int \frac{|s+7+K|^{\frac{1}{6}}|7s+1+K|^{\frac{1}{6}}(2s+2-K)^{\frac{2}{3}}}{3s(s-1)} \bigg|_{K=\sqrt{s^2+14s+1}} \mathrm{d}s \,\bigg|_{s=\omega^3}.$$

The corresponding solution of the original equation (1.1) is

- $$u=\int\frac{|s+7+K|^{\frac16}|7s+1+K|^{\frac16}(2s+2-K)^{\frac23}}{3s(s-1)}\Big|_{K=\sqrt{s^2+14s+1}}\mathrm{d}s\,\Big|_{s=\frac{y^3}{x^3}}.$$

For the general value of κ, $\kappa\neq0,1,2$, all Lie and discrete point symmetries of the associated equation $\varphi_{\omega\omega\omega}=-N/M$ are symmetries of the system $M=N=0$. This is why the maximal Lie invariance algebra $\mathfrak{a}^{\kappa}_{2.1}$ of reduced equation 2.1^{κ} is equal to $\langle\varphi\partial_\varphi\rangle$, and thus it is entirely induced by $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}^{\kappa}_{2.1})$.

We compute the point symmetry group $G^{\kappa}_{2.1}$ of reduced equation 2.1^{κ} with an arbitrary nonsingular value $\kappa\neq0,1,2$ by the algebraic method. Let Φ: $\tilde\omega=\Omega(\omega,\varphi)$, $\tilde\varphi=F(\omega,\varphi)$ with $\Omega_\omega F_\varphi-\Omega_\varphi F_\omega\neq0$ be a point symmetry transformation of this equation. From the condition $\Phi_*\mathfrak{a}_{2.1}\subseteq\mathfrak{a}_{2.1}$, we only derive the equations $\Omega_\varphi=0$ and $\varphi F_\varphi=aF$, which mean that $\Omega=\Omega(\omega)$ with $\Omega_\omega\neq0$ and $F=g(\omega)\varphi^a+f(\omega)$ with a nonzero constant a, a nonvanishing function g of ω and a function f of ω. The further computation by the direct method is the most complicated among such computations in this chapter. The left-hand side $L[\varphi]$ of reduced equation 2.1^{κ} is a homogeneous second-degree polynomial with respect to the unknown function φ and its derivatives with coefficients depending on ω. After expanding the transformed equation with taking into account the obtained form of Φ and collecting the coefficients of ${\varphi_{\omega\omega}}^3\varphi_\omega$, we first derive the equation $a=1$, which means that the transformation Φ is affine with respect to φ. Then the condition of preserving reduced equation 2.1^{κ} by Φ can be written in the form $L[\Phi_*\varphi]=K(\omega)L[\varphi]$, where K is a nonvanishing function of ω. Collecting, in the last equality, coefficients of the terms that are of degree two with respect to the unknown function φ and its derivatives leads to a system of determining equations for the functions Ω, g and K, whose general solution consists of two families, $(\Omega,g,K)=(\omega,c_1,c_1^2)$ and $(\Omega,g,K)=(\omega^{-1},c_1\omega^{-\kappa},c_1^2\omega^{-2\kappa-11})$, where c_1 is an arbitrary nonzero constant. For each of the found solutions for

(Ω, g, K), the system for f derived by collecting coefficients of the remaining terms only has the zero solution. As a result, for any value of κ the entire group $G^{\kappa}_{2.1}$ is induced by the stabilizer of $\mathfrak{s}^{\lambda}_{2.1}$ in G with the corresponding value of λ.

Polynomial solutions of reduced equations 2.1^{κ} whose degree is not greater than five and that result in nonpolynomial solutions of the original equation (1.1) are exhausted by $\varphi = c\omega$ for $\kappa = 5/2$ and $\varphi = c(\omega^3 - 8/21)$ for $\kappa = 9/2$, where $c = 1$ modulo the induced $G_{1.1}$-equivalence. The first solution corresponds to the solution $u = |x|^{3/2}y$ of (1.1), which can also be obtained and, moreover, generalized using the multiplicative separation of variables; make the permutation $\mathcal{J}$ of x and y in the last solution of Section 2.7. The first solution gives a new solution of (1.1),

- $u = |x|^{9/2}\left(\dfrac{y^3}{x^3} - \dfrac{8}{21}\right).$

2.2. $\mathfrak{s}^{\nu}_{2.2} = \big\langle D^t(1),\ D^t(t) - \frac13 D^{\rm s} + P^x(1) + P^y(\nu)\big\rangle,\ \ |\nu| \leqslant 1 \ (\mathrm{mod}\ G)$:

$$u = \mathrm{e}^{-x}\varphi, \quad \omega = y - \nu x;$$
$$\big(2\nu(\nu^3-1)\varphi_{\omega\omega} + (3\nu^3-1)\varphi_\omega + \nu^2\varphi\big)\varphi_{\omega\omega\omega} + (5\nu^3-1){\varphi_{\omega\omega}}^2 + \nu(11\nu\varphi_\omega + 3\varphi)\varphi_{\omega\omega} + 5\nu\varphi_\omega^{\,2} + 2\varphi\varphi_\omega = 0.$$

The associated system $M = N = 0$ (see the beginning of this section) is equivalent to the equation $\varphi_\omega = 0$ if $\nu = 0$, $2\varphi_\omega = \varphi$ if $\nu = -1$ or $\varphi = 0$ otherwise; the corresponding solutions of the original equation (1.1) belong to the family of trivial solutions (2.1) or to the family (2.4). All Lie and discrete point symmetries of the associated equation $\varphi_{\omega\omega\omega} = -N/M$ are symmetries of the system $M = N = 0$. This is why for any value of ν, the maximal Lie invariance algebra of reduced equation 2.2^{ν} is the algebra $\mathfrak{a}_{2.2} = \langle \partial_\omega, \varphi\partial_\varphi\rangle$, and this equation is invariant with respect to the group $G_{2.2}$, which consists of the point transformations $\tilde\omega = \omega + \tilde c_1$, $\tilde\varphi = \tilde c_2\varphi$, where $\tilde c_1$ and $\tilde c_2$ are arbitrary constants with $\tilde c_2 \neq 0$. All the subalgebras $\mathfrak{s}^{\nu}_{2.2}$ have the same normalizer $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}^{\nu}_{2.2}) = \langle D^t(1), D^t(t) - \frac13 D^{\rm s}, P^x(1), P^y(1)\rangle$ in $\mathfrak{g}$.

The vector fields $D^t(1)$, $D^t(t)-\frac{1}{3}D^{\rm s}$, $P^x(1)$ and $P^y(1)$ from $\mathrm{N}_{\mathfrak g}(\mathfrak s^\nu_{2.2})$ induce the Lie-symmetry vector fields 0, $-\varphi\partial_\varphi$, $-\nu\partial_\omega+\varphi\partial_\varphi$ and ∂_ω of reduced equation 2.2^ν, respectively, and thus the algebra $\mathfrak a_{2.2}$ is entirely induced by elements of $\mathrm{N}_{\mathfrak g}(\mathfrak s^\nu_{2.2})$. The discrete point symmetry transformation $\mathcal J^{\rm i}\circ\mathcal J^{\rm s}$: $(\tilde t,\tilde x,\tilde y,\tilde u)=(-t,x,y,-u)$ of (1.1) induces the discrete point symmetry transformation $(\tilde\omega,\tilde\varphi)=(\omega,-\varphi)$ for any of reduced equations 2.2^ν. Therefore, the entire group $G_{2.2}$ is induced by the point symmetry group G of the original equation (1.1).

We construct the point symmetry group $G^\nu_{2.2}$ of reduced equation 2.2^ν with an arbitrary fixed ν using the algebraic method. Let Φ: $\tilde\omega=\Omega(\omega,\varphi)$, $\tilde\varphi=F(\omega,\varphi)$ with $\Omega_\omega F_\varphi-\Omega_\varphi F_\omega\neq0$ be a point symmetry transformation of this equation. The necessary condition $\Phi_*\mathfrak a_{2.2}\subseteq\mathfrak a_{2.2}$ implies the equations $\Omega_\omega=a_{11}$, $\varphi\Omega_\varphi=a_{21}$, $F_\omega=a_{12}F$ and $\varphi F_\varphi=a_{22}F$, where a_{11}, a_{12}, a_{21} and a_{22} are constants with $a_{11}a_{22}-a_{12}a_{21}\neq0$. Therefore,

$$\Omega=a_{11}\omega+a_{21}\ln|\varphi|+\tilde c_1,\quad F=\tilde c_2{\rm e}^{a_{12}\omega}\varphi^{a_{22}},$$

where $\tilde c_1$ and $\tilde c_2$ are arbitrary constants with $\tilde c_2\neq0$. We continue the computation with the direct method using the derived form for Φ, which leads to a cumbersome overdetermined system of determining equations for the parameters a_{11}, a_{12}, a_{21} and a_{22}, whose solution depends on the value of ν. For $\nu\neq\pm1$, we obtain the single solution $a_{12}=a_{21}=0$, $a_{11}=a_{22}=1$, i.e., the complete point symmetry group $G^\nu_{2.2}$ of reduced equation 2.2^ν with such values of ν coincides with the common point symmetry group $G_{2.2}$. For each $\nu\in\{-1,1\}$, there is exactly one more solution $a_{11}=-1$, $a_{21}=0$, $a_{12}=\nu$, $a_{22}=1$, i.e., in addition to the elements of $G_{2.2}$, the complete point symmetry group $G^\nu_{2.2}$ of reduced equation 2.2^ν with $\nu=\pm1$ contains the transformations $\tilde\omega=-\omega+\tilde c_1$, $\tilde\varphi=\tilde c_2{\rm e}^{\nu\omega}\varphi$, where $\tilde c_1$ and $\tilde c_2$ are again arbitrary constants with $\tilde c_2\neq0$. This means that the group $G^\nu_{2.2}$ with $\nu=1$ or $\nu=-1$ is generated by the elements of $G_{2.2}$ and the discrete point symmetry transformation $(\tilde\omega,\tilde\varphi)=(-\omega,{\rm e}^{\nu\omega}\varphi)$, which is induced by the discrete point symmetry $\mathscr J$ or $\mathscr J\circ\mathcal J^{\rm s}$ of the original equation (1.1), respectively. Therefore, for any value of ν the group $G^{\nu}_{2.2}$ is entirely induced by the stabilizer of $\mathfrak{s}^{\nu}_{2.2}$ in G.

Up to the $G_{\rm L}$-equivalence, the subalgebra $\mathfrak{s}^{\nu}_{2.2}$ with any value of ν is prolonged in a unique way to v, which leads to the subalgebra

$$\bar{\mathfrak{s}}^{\nu}_{2.2} = \big\langle \bar{D}^t(1),\ \bar{D}^t(t) - \tfrac{1}{3}\bar{D}^{\rm s} + \bar{P}^x(1) + \bar{P}^y(\nu)\big\rangle$$

of the algebra $\mathfrak{g}_{\rm L}$. Therefore, up to the $G_{\rm L}$-equivalence, there is a unique extension $v = {\rm e}^{-x/2}\psi$ of ansatz 2.2 to v, and the corresponding reduced system is

$$\begin{aligned}&24\psi_\omega\varphi_{\omega\omega} - 12(2\nu\psi_\omega + \psi)(\nu\varphi_\omega + \varphi) + 8(2\nu^3+1){\psi_\omega}^3\\ &\qquad + 12\nu^2\psi{\psi_\omega}^2 - \psi^3 = 0,\\ &\nu\varphi_{\omega\omega} + \varphi_\omega + \nu{\psi_\omega}^2 + \frac{1}{2}\psi\psi_\omega = 0.\end{aligned}$$

It is obvious that an arbitrary function of the form $\varphi = c_1{\rm e}^{-\omega/\nu}+c_2$ if $\nu \neq 0$ or an arbitrary constant if $\nu = 0$ is a solution of reduced equation 2.2^{ν}, and these solutions lead to trivial solutions of the equation (1.1) from the family (2.1). Further we ignore the above trivial solutions.

Reduced equation 2.2^0 is especially short, $\varphi_\omega\varphi_{\omega\omega\omega} + {\varphi_{\omega\omega}}^2 = 2\varphi\varphi_\omega$, and integrates twice to ${\varphi_\omega}^3 = \varphi^3 + c_1\varphi + c_2$, where c_1 and c_2 are the integration constants. Separating the variables and integrating further, we construct the general solution of reduced equation 2.2^0 in an implicit form with one quadrature,

$$\int \frac{{\rm d}\varphi}{(\varphi^3 + c_1\varphi + c_2)^{1/3}} = \omega + c_3, \tag{2.16}$$

where c_3 is one more integration constant. The corresponding solutions of the original equation (1.1) are of the form

- $u = {\rm e}^{-x}\varphi(y),$ (2.17)

where the function $\varphi = \varphi(y)$ is implicitly defined by (2.16), where $\omega = y$, $c_3 = 0 \pmod G$, and, up to the G-equivalence, the constant c_1, if it is nonzero, can be set to be equal ± 1 or the constant c_2, if it is nonzero, can be set to be equal one. The solution family (2.17) can be extended using the multiplicative separation of variables, see Section 2.7. The formula (2.16) obviously leads to explicit solutions of reduced equation 2.2^0 only if $c_1 = c_2 = 0$, which gives $\varphi = \tilde{c}_3 \mathrm{e}^{\omega}$ with an arbitrary constant $\tilde{c}_3$. All the corresponding solutions of the equation (1.1), $u = \tilde{c}_3 \mathrm{e}^{y-x}$, belong to the family of simple solutions (2.4).

For several specific values of (c_1, c_2), when the integral in the left-hand side of (2.16) is reduced to cases of the Chebyshev theorem on the integration of binomial differentials, it be expressed in terms of elementary functions. This gives the following $G_{2.2}$-inequivalent parametric solutions (without quadratures) of reduced equation 2.2^0:

$$\circ \quad \varphi = \frac{s^3+2}{s^3-1} \quad \text{with} \quad \frac{1}{2}\ln\frac{s^2+s+1}{(s-1)^2} - \sqrt{3}\arctan\frac{2s+1}{\sqrt{3}} = y,$$

$$\circ \quad \varphi = |s^3-1|^{-1/2}$$
$$\text{with} \quad \frac{1}{2}\ln\frac{s^2+s+1}{(s-1)^2} - \sqrt{3}\arctan\frac{2s+1}{\sqrt{3}} = 2y, \tag{2.18}$$

$$\circ \quad \varphi = (s^3-1)^{-1/3} \quad \text{with} \quad \frac{1}{2}\ln\frac{s^2+s+1}{(s-1)^2} - \sqrt{3}\arctan\frac{2s+1}{\sqrt{3}} = 3y$$

if $4c_1^3 = -27c_2^2$, $(c_1 \neq 0,\ c_2 = 0)$ and $(c_1 = 0,\ c_2 \neq 0)$, respectively; cf. [92, Section 4.1.1.2], where there are several typos and a needless involvement of complex numbers. In the first case, the polynomial $\varphi^3 + c_1\varphi + c_2$ has a root λ of multiplicity two and can thus be factorized to $(\varphi+2\lambda)(\varphi-\lambda)^2$, i.e., $c_1 = -3\lambda^2$ and $c_2 = 2\lambda^3$. It is then obvious that we can set $\lambda = 1$ up to $G_{2.2}$-equivalence, more precisely, by scaling of φ. In the second and the third cases, we analogously can set $c_1 = \operatorname{sgn}(s^3-1)$ and $c_2 = 1$, respectively. For $c_1 \neq 0$, the integral in the left-hand side of (2.16) was reduced in [92, Eqs. (33)–(34)] to an integral that, as stated therein, can be expressed in terms of elliptic functions but the corresponding representation of the solution (2.16) does not seem useful.

For general values of ν, the order of reduced equation 2.2^ν can be lowered by the differential substitution $z=\varphi_\omega/\varphi$, $p=\varphi_{\omega\omega}/\varphi$ inspired by the Lie invariance algebra $\mathfrak{a}_{2.2}$. This leads to a first-order ordinary differential equation with respect to $p=p(z)$,

$$\begin{aligned}&\big(2\nu(\nu^3-1)p+3\nu^3z-z+\nu^2\big)(p-z^2)p_z\\&\quad+\big(2\nu(\nu^3-1)z+5\nu^3-1\big)p^2\\&\quad+\big((3\nu^3-1)z^2+12\nu^2z+3\nu\big)p+5\nu z^2+2z=0.\end{aligned}\tag{2.19}$$

Looking for solutions of (2.19) that are at most quadratic with respect to z, we construct only the solutions $p=-z/\nu$ if $\nu=0$, $p=4$ if $\nu=1/2$ and $p=z-1$ if $\nu=-1$. For reduced equation 2.2^ν, this gives the above trivial solutions $\varphi=c_1\mathrm{e}^{-\omega/\nu}+c_2$ if $\nu=0$ as well as $\varphi=c_1\mathrm{e}^{-2\omega}+c_2\mathrm{e}^{2\omega}$ if $\nu=1/2$ and $\varphi=c_1\mathrm{e}^{\omega/2}\cos(\frac12\sqrt3\omega+c_2)$ if $\nu=-1$. As a result, up to G-equivalence we construct the following new solutions of the original equation (1.1):

- $u=\mathrm{e}^{y-x}\pm\mathrm{e}^{-y}$,
- $u=\mathrm{e}^{y-x}\cos\big(\sqrt3(x+y)\big)$.

If $\nu=1$, the equation (2.19) becomes the Abel equation of the second kind,

$$(2z+1)(p-z^2)p_z+4p^2+(2z^2+12z+3)p+5z^2+2z=0,$$

which is reduced by the point transformation $s=z+\frac12$, $r=(z+\frac12)^2(p-z^2)$ to the simpler Abel equation

$$16rr_s+4s(28s^2-1)r+s^3(4s^2-1)(12s^2+1)=0,$$

whose general solution in implicit form is

$$\frac{(4r+4s^4-s^2)^2\big((144r+144s^4-1)^2-(12s^2+1)^3\big)}{\Big(3\big(32r+(8s^2+1)(4s^2-1)\big)^2-(8s^2+1)(4s^2-1)^2\Big)^2}=c_1.$$

The latter equation has two polynomial solutions up to degree four,

$$r=-\frac14s^2(4s^2-1)\quad\text{and}\quad r=-\frac1{64}(4s^2-1)(12s^2+1),$$

which correspond to the values $c_1 = 0$ and $c_1 = 1/256$ and the solutions

$$p = -z \quad \text{and} \quad p = \frac{z(z^3 - 2z^2 - 3z - 1)}{(2z+1)^2}$$

of the former Abel equation, respectively. After the inverse differential substitution, we respectively obtain two ordinary differential equations. The integration of the first one only results in trivial solutions of the original equation (1.1), whereas solving the second equation, we construct, up to the G-equivalence, the following parametric solution of (1.1):

- $$u = \mathrm{e}^{-(x+y)/2}\frac{(3z^2+3z+1)^{-1/6}}{|z|^{1/2}|z+1|^{1/2}}$$
  $$\text{with} \quad \ln\left|\frac{z+1}{z}\right| - \frac{2}{\sqrt{3}}\arctan\big(\sqrt{3}(2z+1)\big) = y - x.$$

2.3. $\tilde{\mathfrak{s}}^{\nu}_{2.3} = \big\langle D^t(1),\ 2D^t(t) + \frac{1}{3}D^{\mathrm{s}} + R^x(1) + R^y(\nu)\big\rangle$, $|\nu| \leqslant 1 \pmod G$ (we replace the subalgebra $\mathfrak{s}^{\nu}_{2.3}$ by the G-equivalent subalgebra $\tilde{\mathfrak{s}}^{\nu}_{2.3}$ for convenience of the reduction procedure):

$$u = x\varphi + (y+\nu x)\ln|x|, \quad \omega = y/x;$$
$$\big(2\omega(\omega^3-1)\varphi_{\omega\omega} - 2\omega^3 + \nu\omega^2 + 1\big)\varphi_{\omega\omega\omega} + (5\omega^3-1){\varphi_{\omega\omega}}^2$$
$$- \omega(8\omega - 3\nu)\varphi_{\omega\omega} + 3\omega - 2\nu = 0.$$

For any values of ν, reduced equation 2.3^ν can be represented in normal form since the coefficient of $\varphi_{\omega\omega\omega}$ in it does not vanish on its solutions. Its maximal Lie invariance algebra is the algebra $\mathfrak{a}_{2.3} = \langle \partial_\varphi, \omega\partial_\varphi\rangle$. The corresponding Lie group $G_{2.3}$ consists of the point transformations $\tilde{\omega} = \omega$, $\tilde{\varphi} = \varphi + \tilde{c}_1\omega + \tilde{c}_2$, where $\tilde{c}_1$ and $\tilde{c}_2$ are arbitrary constants. All the subalgebras $\tilde{\mathfrak{s}}^{\nu}_{2.3}$ have the same normalizer $\mathrm{N}_{\mathfrak{g}}(\tilde{\mathfrak{s}}^{\nu}_{2.3}) = \langle D^t(1), 2D^t(t) + \frac{1}{3}D^{\mathrm{s}}, R^x(1), R^y(1)\rangle$ in $\mathfrak{g}$. The vector fields $D^t(1)$, $2D^t(t) + \frac{1}{3}D^{\mathrm{s}}$, $R^x(1)$ and $R^y(1)$ from $\mathrm{N}_{\mathfrak{g}}(\tilde{\mathfrak{s}}^{\nu}_{2.3})$ induce the Lie-symmetry vector fields 0, $-(\omega+\nu)\partial_\varphi$, ∂_φ and $\omega\partial_\varphi$ of reduced equation 2.3^ν, respectively, i.e., the algebra $\mathfrak{a}_{2.3}$ is entirely induced, and thus the entire group $G_{2.3}$ is induced by the stabilizer of $\tilde{\mathfrak{s}}^{\nu}_{2.3}$ in G.

Using the algebraic method, we compute the point symmetry group $G^\nu_{2.3}$ of reduced equation 2.3^ν for any fixed value of ν. Let Φ: $\tilde\omega=\Omega(\omega,\varphi)$, $\tilde\varphi=F(\omega,\varphi)$ with $\Omega_\omega F_\varphi-\Omega_\varphi F_\omega\neq0$ be a point symmetry transformation of this equation. The condition $\Phi_*\mathfrak a_{2.3}\subseteq\mathfrak a_{2.3}$, implies the equations $\Omega_\varphi=0$, $F_\varphi=a_{11}+a_{12}\Omega$ and $\omega F_\varphi=a_{21}+a_{22}\Omega$, where a_{11}, a_{12}, a_{21} and a_{22} are constants with $a_{11}a_{22}-a_{12}a_{21}\neq0$. Therefore,

$$\Omega=\frac{-a_{11}\omega+a_{21}}{a_{12}\omega-a_{22}},\quad F=-\frac{a_{11}a_{22}-a_{12}a_{21}}{a_{12}\omega-a_{22}}\varphi+f(\omega)$$

with a function f of ω. Taking into account the derived form for Φ, we continue the computation with the direct method. As a result, we obtain a cumbersome overdetermined system of determining equations for the parameters a_{11}, a_{12}, a_{21}, a_{22} and f, whose solution depends on the value of ν. For $\nu\neq\pm1$, we obtain that $a_{12}=a_{21}=0$, $a_{11}=a_{22}=1$ and $f=\tilde c_1\omega+\tilde c_2$ with arbitrary constants $\tilde c_1$ and $\tilde c_2$. In other words, the complete point symmetry group $G^\nu_{2.3}$ of reduced equation 2.3^ν with $\nu\neq\pm1$ coincides with the common Lie symmetry group $G_{2.3}$. For $\nu=\pm1$, we have additional solutions,

$$a_{11}=a_{22}=0,\quad a_{12}=a_{21}=\nu,\quad f=-(\nu+\omega^{-1})\ln|\omega|+\tilde c_1\omega^{-1}+\tilde c_2,$$

where $\tilde c_1$ and $\tilde c_2$ are again arbitrary constants. Hence the complete point symmetry group $G^\nu_{2.3}$ of reduced equation 2.3^ν with $\nu=\pm1$ is generated by the elements of the common Lie symmetry group $G_{2.3}$ and the discrete point symmetry transformation

$$\tilde\omega=\omega^{-1},\quad \tilde\varphi=\nu\omega^{-1}\varphi-(\nu+\omega^{-1})\ln|\omega|,$$

which is induced by the discrete point symmetries $\mathscr J$ and $\mathscr J\circ\mathscr J^{\rm s}$ of the original equation (1.1) if $\nu=1$ and $\nu=-1$, respectively. Therefore, for any value of ν the group $G^\nu_{2.3}$ is entirely induced by the stabilizer of $\tilde{\mathfrak s}^\nu_{2.3}$ in G.

For each value of ν, the subalgebra $\tilde{\mathfrak s}^\nu_{2.3}$ has, up to the $G_{\rm L}$-equivalence, a single counterpart among subalgebras of the algebra $\mathfrak g_{\rm L}$,

$$\bar{\mathfrak s}^\nu_{2.3}=\big\langle\bar D^t(1),\,2\bar D^t(t)+\tfrac13\bar D^{\rm s}+\bar R^x(1)+\bar R^y(\nu)\big\rangle.$$

This is why ansatz 2.3 is extended to v in a unique way up to the $G_{\rm L}$-equivalence as $v=|x|^{1/2}\psi$, and the nonlinear Lax representation (1.14) reduces to the system

$$\begin{aligned}&16\varepsilon\omega(\omega^3-1)\psi_\omega{}^3-12\varepsilon(\omega^3-1)\psi\psi_\omega{}^2-24(\nu\omega^2-1)\psi\psi_\omega\\&\qquad+\varepsilon\omega\psi^3+12\nu\omega\psi=0,\\&\omega\varphi_{\omega\omega}+\varepsilon\omega\psi_\omega{}^2-\frac{\varepsilon}{2}\psi\psi_\omega-1=0,\end{aligned}\tag{2.20}$$

where $\varepsilon:=\operatorname{sgn}x$.

Reduced equation 2.3^ν is an Abel equation of the second kind with respect to $\varphi_{\omega\omega}$. Its particular solution $\varphi=\omega\ln|\omega|$ corresponds to a solution of the system (2.20) with $\psi=0$ and the trivial solution $u=y\ln|y|+\nu x\ln|x|$ of (1.1) from the family (2.1). The differential substitution $\varphi_{\omega\omega}=p+\omega^{-1}$ maps reduced equation 2.3^ν to the simpler Abel equation of the second kind $\big(2\omega(\omega^3-1)p+\nu\omega^2-1\big)p_\omega+(5\omega^3-1)p^2+3\nu\omega p=0$.

2.4. $\tilde{\mathfrak s}_{2.4}=\big\langle D^t(1),\,3D^t(t)+Z(1)\big\rangle$ (we replace the subalgebra $\mathfrak s_{2.4}$ by the G-equivalent subalgebra $\tilde{\mathfrak s}_{2.4}$ for convenience of the reduction procedure):

$$u=\varphi+\ln|x|,\quad \omega=y/x;$$

$$\begin{aligned}&\big(2\omega(\omega^3-1)\varphi_{\omega\omega}+(3\omega^3-1)\varphi_\omega-\omega^2\big)\varphi_{\omega\omega\omega}\\&\qquad+2(5\omega^3-1)\varphi_{\omega\omega}{}^2+2\omega(11\omega\varphi_\omega-3)\varphi_{\omega\omega}+2(5\omega\varphi_\omega-2)\varphi_\omega=0.\end{aligned}$$

The normalizer of the subalgebra $\tilde{\mathfrak s}_{2.4}$ in $\mathfrak g$ is $\mathrm N_{\mathfrak g}(\tilde{\mathfrak s}_{2.4})=\langle D^t(1),D^t(t),Z(1)\rangle$. The Lie-symmetry vector fields $D^t(1)$, $3D^t(t)+Z(1)$ and $Z(1)$ of the equation (1.1) induce the Lie-symmetry vector fields 0, 0 and ∂_φ of reduced equation 2.4, respectively. This equation can be represented in normal form since the coefficient of $\varphi_{\omega\omega\omega}$ in it does not vanish on its solutions. The maximal Lie invariance algebra $\mathfrak a_{2.4}$ of reduced equation 2.4 is one-dimensional, $\mathfrak a_{2.4}=\langle\partial_\varphi\rangle$, and thus it is entirely induced by $\mathrm N_{\mathfrak g}(\tilde{\mathfrak s}_{2.4})$.

We compute the point symmetry group $G_{2.4}$ of reduced equation 2.4 by the algebraic method. Let Φ: $\tilde\omega=\Omega(\omega,\varphi)$, $\tilde\varphi=F(\omega,\varphi)$ with $\Omega_\omega F_\varphi-$

$\Omega_\varphi F_\omega \neq 0$ be a point symmetry transformation of this equation. From the condition $\Phi_*\mathfrak{a}_{2.4} \subseteq \mathfrak{a}_{2.4}$, we derive the equations $\Omega_\varphi = 0$ and $F_{\varphi\varphi} = 0$, which mean that $\Omega = \Omega(\omega)$ and $F = a\varphi + f(\omega)$ with $\Omega_\omega \neq 0$, a nonzero constant a and a function f of ω. Taking into account the derived form for Φ, we continue the computation with the direct method and obtain a cumbersome overdetermined system of determining equations for the parameters Ω, a and f, which can nevertheless be solved, giving $a = 1$ and either $\Omega = \omega$ and $f = c$ or $\Omega = \omega^{-1}$ and $f = \ln|\omega| + c$ with an arbitrary constant c. Therefore, the group $G_{2.4}$ is generated by the one-parameter subgroup of the shifts with respect to φ and the discrete point symmetry transformation $\tilde\omega = \omega^{-1}$, $\tilde\varphi = \varphi + \ln|\omega|$. The last transformation is induced by the permutation $\mathcal{J}$ of the variables x and y in the original equation (1.1). Therefore, the group $G_{2.4}$ is entirely induced by the stabilizer of $\tilde{\mathfrak{s}}^{\nu}_{2.4}$ in G.

The subalgebra $\tilde{\mathfrak{s}}_{2.4}$ has the family of $G_{\rm L}$-inequivalent counterparts $\bar{\mathfrak{s}}^{\varepsilon\nu}_{2.4} = \big\langle \bar D^t(1),\, 3\bar D^t(t) + \bar Z(\varepsilon) + \nu\bar P^v \big\rangle$ among subalgebras of the algebra $\mathfrak{g}_{\rm L}$. Here $\varepsilon = \pm1$ and $\nu \geqslant 0 \pmod{G_{\rm L}}$.[2.3] This results in a family of $G_{\rm L}$-inequivalent extensions of ansatz 2.4 to v that are parameterized by ε and ν, $u = \varphi + \varepsilon\ln|x|$, $v = \psi + \nu\ln|x|$. The corresponding reduced systems are

$$
\begin{aligned}
&3\big((\omega^3+1)\psi_\omega - \nu\omega^2\big)\varphi_\omega - 2\omega(\omega^3-1)\psi_\omega{}^3 + 3\nu(\omega^3-1)\psi_\omega{}^2\\
&\qquad - 3\varepsilon\omega\psi_\omega - \nu(\nu^2-3\varepsilon)\omega = 0,\\
&\omega\varphi_{\omega\omega} + \varphi_\omega + \omega\psi_\omega{}^2 - \nu\psi_\omega = 0.
\end{aligned}
\tag{2.21}
$$

The condition of vanishing the coefficient of φ_ω in the first equation of (2.21) is consistent with (2.21) only if $\nu = 0$ and thus $\psi_\omega = 0$, which implies in view of the second equation of (2.21) that $\omega\varphi_{\omega\omega} + \varphi_\omega = 0$. The associated family of particular solutions $\varphi = c_1\ln|\omega| + c_2$ of reduced equation 2.4, which are parameterized by the arbitrary con-

[2.3] The discrete point symmetry $\mathcal{J}^{\rm s} := \mathcal{D}^{\rm s}(-1)$ of the equation (1.1), which alternates the signs of (x, y, u) and of the vector fields $Z(\sigma)$ from the algebra $\mathfrak{g}$, has no counterpart among point symmetries of the nonlinear Lax representation (1.14). As a result, in contrast to the subalgebra $\tilde{\mathfrak{s}}_{2.4}$, the parameter ε in the subalgebra family $\{\bar{\mathfrak{s}}^{\varepsilon\nu}_{2.4}\}$ can be gauged, up to the $G_{\rm L}$-equivalence, merely to ±1 but not to 1.

stants c_1 and c_2, corresponds to the subfamily of the trivial solutions $u = c_1 \ln|y| + (c_1+1)\ln|x| + c_2$ from the family (2.1).

Further $(\omega^3+1)\psi_\omega - \nu\omega^2 \neq 0$. Solving the first equation of (2.21) with respect to φ_ω and excluding φ from the second equation of (2.21), we derive a first-order ordinary differential equation with respect to $\zeta := \psi_\omega$,

$$\begin{aligned}\big(4\omega(\omega^6-1)\zeta^3 - 3\nu(\omega^3-1)(3\omega^3+1)\zeta^2 + 6\nu^2\omega^2(\omega^3-1)\zeta \\ - \nu\omega(\nu^2\omega^3+\nu^2-3\varepsilon)\big)\zeta_\omega + (7\omega^6+18\omega^3-1)\zeta^4 \\ - 6\nu\omega^2(3\omega^3+5)\zeta^3 + 3\omega(5\nu^2\omega^3+3\nu^2+3\varepsilon)\zeta^2 \\ - 2\nu(2\nu^2\omega^3-\nu^2+3\varepsilon)\zeta = 0.\end{aligned} \tag{2.22}$$

Let $\nu = 0$ and thus $\zeta \neq 0$. Then the equation (2.22) reduces to the simple Bernoulli equation

$$4\omega(\omega^6-1)\zeta\zeta_\omega + (7\omega^6+18\omega^3-1)\zeta^2 + 9\varepsilon\omega = 0,$$

which integrates to $\zeta = \pm(\omega^3-1)^{-1}\sqrt{\tilde{c}_1|\omega|^{-1/2}(\omega^3+1)+3\varepsilon\omega}$. The first equation of (2.21) with this value of ζ implies that

$$\varphi = \frac{\varepsilon}{3}\ln|\omega^3-1| + c_1\ln\left|\frac{|\omega|^{3/2}+1}{|\omega|^{3/2}-1}\right| + c_2 \quad \text{for} \quad \omega \geqslant 0,$$
$$\varphi = \frac{\varepsilon}{3}\ln|\omega^3-1| + c_1\arctan|\omega|^{3/2} + c_2 \quad \text{for} \quad \omega \leqslant 0,$$

where c_1 and c_2 are arbitrary constants, and the constants c_2 and ε can be set to 0 and 1 up to the $G_{2.4}$-equivalence, respectively. The corresponding solutions of the equation (1.1) are

- $u = \dfrac{1}{3}\ln|y^3-x^3| + c_1\ln\left|\dfrac{|x|^{3/2}+|y|^{3/2}}{|x|^{3/2}-|y|^{3/2}}\right| \quad \text{for} \quad xy \geqslant 0,$
- $u = \dfrac{1}{3}\ln|y^3-x^3| + c_1\arctan\left|\dfrac{y}{x}\right|^{3/2} \quad \text{for} \quad xy \leqslant 0.$

For $\nu = \sqrt{3}$, the equation (2.22) can be integrated implicitly,

$$4\ln\big|(\omega^3-1)\zeta-\sqrt{3}\omega^2\big| - 2\ln\big|(\omega^3+1)\zeta-\sqrt{3}\omega^2\big| + \ln\big|\omega\zeta-\sqrt{3}\big| + \ln\big|\zeta\big| = c_1.$$

Then the function φ is defined by the first equation of (2.21) with $\psi_\omega = \zeta$ and $\nu = \sqrt{3}$.

2.13. $\mathfrak{s}_{2.13} = \big\langle D^t(1),\, D^{\rm s}\big\rangle$: $\quad u = x^3\varphi, \quad \omega = y/x;$

$$\big(2\omega(\omega^3-1)\varphi_{\omega\omega} - 2(3\omega^3-1)\varphi_\omega + 6\omega^2\varphi\big)\varphi_{\omega\omega\omega} \\ - (5\omega^3-1)\varphi_{\omega\omega}^2 + 2\omega(11\omega\varphi_\omega - 9\varphi)\varphi_{\omega\omega} - 4(5\omega\varphi_\omega - 6\varphi)\varphi_\omega = 0.$$

The normalizer of the subalgebra $\mathfrak{s}_{2.13}$ in $\mathfrak{g}$ is $\mathrm{N}_\mathfrak{g}(\mathfrak{s}_{2.13}) = \langle D^t(1), D^t(t), D^{\rm s}\rangle$, and the entire maximal Lie invariance algebra $\mathfrak{a}_{2.13} = \langle\varphi\partial_\varphi\rangle$ of reduced equation 2.13 is induced by this normalizer.

Reduced equation 2.13 can be included in the family of reduced equations 2.1^κ as the element with $\kappa = 3$, which corresponds to the limit values $\lambda = \pm\infty$.

Simple solutions of reduced equation 2.13, $\varphi = |\omega|^{3/2}$ and the family of solutions that are cubic polynomials in ω, were found in [92], see the equations (24) and (25) therein, respectively. The solution $u = |xy|^{3/2}$ of the original equation (1.1), which corresponds to the solution $\varphi = |\omega|^{3/2}$, is essentially generalized in Section 2.7 using multiplicative separation of the variables x and y. The above family of polynomial solutions in ω is associated with the family of solutions of (1.1) that are homogeneous cubic polynomials in (x, y) with constant coefficients.

2.6. Lie reductions to algebraic equations

For any three-dimensional subalgebra $\mathfrak{s}_3$ of the algebra $\mathfrak{g}$, either its rank r is less than three and thus it cannot be used for Lie reduction of the equation (1.1) to an algebraic equation or all the corresponding invariant solutions are, up to the G-equivalence, just particular elements of parameterized families of solutions that have been constructed in Sections 2.4 and 2.5. To show this, we present an outline of the classification of three-dimensional subalgebras of $\mathfrak{g}$.

Consider a three-dimensional subalgebra $\mathfrak{s}_3 = \langle Q^i,\ i = 1,2,3\rangle$ of $\mathfrak{g}$ spanned by three (linearly independent) vector fields

$$Q^i = D^t(\tau^i) + \lambda^i D^{\rm s} + P^x(\chi^i) + P^y(\rho^i) + R^x(\alpha^i) + R^y(\beta^i) + Z(\sigma^i)$$

from $\mathfrak{g}$ with arbitrary smooth functions τ^i, χ^i, ρ^i, α^i, β^i and σ^i of t and arbitrary constants λ^i such that the tuples $(\tau^i, \lambda^i, \chi^i, \rho^i, \alpha^i, \beta^i, \sigma^i)$ are linearly independent. Here and in what follows the index i runs from 1 to 3. The consideration splits into cases mainly depending on two values, $k_1 = k_1(\mathfrak{s}_3) := \dim\langle\tau^i\rangle$ and $k_2 = k_2(\mathfrak{s}_3) := \dim\langle(\tau^i, \lambda^i)\rangle$. For brevity, we use transitions to G-equivalent subalgebras, basis changes and hints from the proofs of Lemmas 2.3 and 2.4 without referring to this. Below, κ_1, κ_2, κ_3 and ν denote constants.

$k_1 = 3$. In view of the classical Lie theorem on Lie algebras of vector fields on the real line,[2.4] we can set $\tau^1 = 1$, $\tau^2 = t$ and $\tau^3 = t^2$, which implies $\mathfrak{s}_3 \subset \mathfrak{g}'$, and thus $\lambda^i = 0$. The vector fields Q^1 and Q^2 reduce to $D^t(1)$ and $D^t(t) + Z(\delta)$ with $\delta \in \{0,1\}$, respectively. We successively derive from the commutation relations $[Q^1, Q^3] = 2Q^2$ and $[Q^2, Q^3] = Q^3$ that $\chi^3, \rho^3, \alpha^3, \beta^3 = \text{const}$, $\sigma^3_t = \delta$ and hence $\chi^3, \rho^3, \alpha^3, \beta^3 = 0$. Therefore, $r = 2$.

$k_1 = 2$, $k_2 = 3$. We can make $\tau^1 = 1$, $\tau^2 = t$, $\tau^3 = 0$, $\lambda^1 = \lambda^2 = 0$, $\lambda^3 = 1$, and then $Q^3 = D^{\rm s}$. The commutation relations $[Q^1, Q^3] = [Q^2, Q^3] = 0$ imply $Q^1 = D^t(1)$ and $Q^2 = D^t(t)$, i.e., $r = 2$.

$k_1 = k_2 = 2$. Setting $Q^1 = D^t(1)$, $\tau^2 = t$, $\tau^3 = 0$, $\lambda^3 = 0$, we derive from the commutation relations $[Q^j, Q^3] = \kappa_j Q^3$, $j = 1,2$, $[Q^1, Q^2] = Q^1 + \kappa_3 Q^3$ that $\chi^3_t = \kappa_1\chi^3$, $t\chi^3_t = \kappa_2\chi^3$, $\rho^3_t = \kappa_1\rho^3$, $t\rho^3_t = \kappa_2\rho^3$, and thus, if $r = 3$, $(\chi^3, \rho^3) \neq (0,0)$, $\kappa_1 = \kappa_2 = 0$, which further implies that $\chi^3_t = \rho^3_t = \alpha_t = \beta_t = \sigma_t = 0$. In other words, the subalgebra $\mathfrak{s}_3$ contains, up to G-equivalence, a subalgebra from the family $\{\mathfrak{s}^{0\nu\delta'}_{2.14}\}$.

[2.4] See [28, 36–38, 77, 78, 98] and references therein for applications of this theorem to classifying subalgebras of various algebras of vector fields.

$k_1 = 1$, $k_2 = 2$. We make $\tau^1 = 1$, $\tau^2 = \tau^3 = 0$, $\lambda^1 = \lambda^3 = 0$, $\lambda^2 = 1$ and then $Q^2 = D^{\rm s}$. The commutation relations $[Q^1, Q^2] = 0$, $[Q^j, Q^3] = \kappa_j Q^3$, $j = 1, 2$, imply $Q^1 = D^t(1)$ and, if $r = 3$, then $\kappa_2 = -1$ and $Q^3 = P^x(\mathrm{e}^{\kappa_1 t})+ \nu P^y(\mathrm{e}^{\kappa_1 t})$, i.e., the subalgebra $\mathfrak{s}_3$ contains a subalgebra that is G-equivalent to one from the family $\{\mathfrak{s}_{2.9}^{\tilde\rho}\}$.

$k_2 \leqslant 1$. If $r = 3$, then up to G-equivalence, the subalgebra $\mathfrak{s}_3$ contains a subalgebra from the family $\{\mathfrak{s}_{2.17}^{\rho\alpha\beta}\}$ and, therefore, a subalgebra from the family $\{\mathfrak{s}_{1.4}^{\beta}\}$.

As a result, we conclude that Lie reductions of the equation (1.1) to algebraic equations give no new G-equivalent solutions in comparison with those that have been constructed in a closed explicit form in Sections 2.4 and 2.5.

2.7. Multiplicative separation of variables

The equation (1.1) is identically satisfied under the additive separation of the variables x and y, and the solutions from the corresponding family (2.1) are trivial.

Consider solutions of the equation (1.1) with nontrivial multiplicative separation of the variables x and y. They are represented in the form $u = \varphi(t, x)\psi(t, y)$ with $\varphi_x \neq 0$ and $\psi_y \neq 0$.

Remark 2.9. The functions φ and ψ are defined up to the transformations $\tilde\varphi = \varphi/f$, $\tilde\psi = f\psi$ with an arbitrary nonzero function of t. If $\varphi_x = 0$ or $\psi_y = 0$, then one can set $\varphi = 1$ or $\psi = 1$, respectively, and thus the separation of the variables x and y is trivial; moreover, then the corresponding solutions belong to the family of trivial solutions (2.1).

Substituting the multiplicative ansatz $u = \varphi(t, x)\psi(t, y)$ into the equation (1.1) and separating the variables x and y, we obtain the equation

$$\frac{\varphi_{tx}}{\varphi_x} + \frac{\psi_{ty}}{\psi_y} = \frac{(\varphi_{xx}\varphi_x)_x}{\varphi_x}\psi + \frac{(\psi_{yy}\psi_y)_y}{\psi_y}\varphi,$$

which we further simultaneously differentiate with respect x and y and derive

$$\frac{1}{\varphi_x}\left(\frac{(\varphi_{xx}\varphi_x)_x}{\varphi_x}\right)_x+\frac{1}{\psi_y}\left(\frac{(\psi_{yy}\psi_y)_y}{\psi_y}\right)_y=0.$$

These two equations imply that

$$\frac{(\varphi_{xx}\varphi_x)_x}{\varphi_x}=\alpha\varphi+\beta,\quad \frac{\varphi_{tx}}{\varphi_x}=\gamma\varphi+\delta\quad\text{and}$$
$$\frac{(\psi_{yy}\psi_y)_y}{\psi_y}=-\alpha\psi+\gamma,\quad \frac{\psi_{ty}}{\psi_y}=\beta\psi-\delta$$

for some sufficiently smooth functions α, β, γ and δ of t. These systems with respect to φ and ψ integrate to

$$\varphi_x{}^3=\frac{\alpha}{2}\varphi^3+\frac{3}{2}\beta\varphi^2+\zeta^1\varphi+\zeta^0,\quad \varphi_t=\frac{\gamma}{2}\varphi^2+\delta\varphi+\zeta^2, \tag{2.23}$$
$$\psi_y{}^3=-\frac{\alpha}{2}\psi^3+\frac{3}{2}\gamma\psi^2+\theta^1\psi+\theta^0,\quad \psi_t=\frac{\beta}{2}\psi^2-\delta\psi+\theta^2, \tag{2.24}$$

where ζ^0, ζ^1, ζ^2, θ^0, θ^1 and θ^2 are also sufficiently smooth functions of t, and for solutions to be nontrivial, we should impose the conditions that the tuples $(\alpha,\beta,\zeta^1,\zeta^0)$ and $(\alpha,\gamma,\theta^1,\theta^0)$ are nonzero. Due to the indeterminacy of (φ,ψ), we set $\delta=0$ without loss of generality.

We exclude the derivatives of the functions φ and ψ in view of the systems (2.23) and (2.24) from their compatibility conditions $(\varphi_x)_t=(\varphi_t)_x$ and $(\psi_y)_t=(\psi_t)_y$ and split the obtained equalities with respect to φ and ψ, which gives the following systems for the parameter functions depending on t:

$$\alpha\beta=\alpha\gamma=\beta\gamma=0,\quad \alpha_t=0,\quad \beta_t=\frac{5}{3}\gamma\zeta^1-\alpha\zeta^2,\quad \gamma_t=\frac{5}{3}\beta\theta^1+\alpha\theta^2,$$
$$\zeta^1_t=3\gamma\zeta^0-3\beta\zeta^2,\quad \theta^1_t=3\beta\theta^0-3\gamma\theta^2,\quad \zeta^0_t=-\zeta^1\zeta^2,\quad \theta^0_t=-\theta^1\theta^2.$$

Consider possible cases separately.

1. $\alpha\neq0$. Then $\alpha=\text{const}$, $\beta=\gamma=\zeta^2=\theta^2=0$, and thus ζ^0, ζ^1, θ^0 and θ^1 are constants. Integrating the systems (2.23) and (2.24) with these

parameters' values and simplifying the result by transformations from G, we derive a family of G-inequivalent solutions of the equation (1.1) that generalizes the solutions (2.17),

- $u=\varphi(x)\psi(y)$,
$$\int\frac{\mathrm{d}\varphi}{(\varphi^3+c_1\varphi+c_2)^{1/3}}=x,\quad \int\frac{\mathrm{d}\psi}{(\psi^3+c_3\psi+c_4)^{1/3}}=-y.$$

Up to the G-equivalence, one of the constants c_1 and c_3, if it is nonzero, can be set to be equal ±1 or one of the constants c_2 and c_4, if it is nonzero, can be set to be equal one. Both quadratures here are the same as in (2.16). Hence they can be computed explicitly for certain values of the tuples (c_1,c_2) and (c_3,c_4), see (2.18).

2. $\beta\neq0$. Then $\alpha=\gamma=0$, and thus $\theta^0=\theta^1=0$, which contradicts the nontriviality condition $\psi_y\neq0$. The case $\gamma\neq0$ reduces to the case $\beta\neq0$ by permutation of x and y.

3. $\alpha=\beta=\gamma=0$, and thus ζ^1 and θ^1 are constants, $\zeta^0=-\int\zeta^1\zeta^2\mathrm{d}t$ and $\theta^0=-\int\theta^1\theta^2\mathrm{d}t$. Rearranging the solution sets of the systems (2.23) and (2.24) with these parameters' values up to the G-equivalence and in view of the indeterminacy of (φ,ψ), we construct the solutions of the equation (1.1) of the form

- $u=\big(|x|^{3/2}+\zeta(t)\big)\big(|y|^{3/2}+\theta(t)\big)$,
- $u=\big(x+\zeta(t)\big)|y|^{3/2}$

and the solution $u=xy$, which belongs to the family (2.7). Here ζ and θ are arbitrary sufficiently smooth functions of t. The first and the second families of solutions generalize the $\mathfrak{s}_{2.13}$-invariant solution $u=|xy|^{3/2}$ and $\mathfrak{s}_{2.1}^{5/3}$-invariant solution $u=|x|^{3/2}y$, see [92, Eq. (26)] and the last paragraph related to reduction 2.1 in Section 2.5.2, respectively.

Remark 2.10. For any ν, the $\mathfrak{s}_{2.2}^{\nu}$-invariant solutions can be interpreted as those with multiplicative separation of variables after their linear change. Following the consideration in this section, one can try to carry out a comprehensive study of such separation of variables. Maybe, the most interesting is the multiplicative separation of the variables $\tilde{x} = x+y$ and $\tilde{y} = x-y$, cf. the last (parametric) solution obtained by reduction 2.2^1.

2.8. Conclusion

In this chapter, we have constructed wide families of new exact invariant solutions of the dispersionless Nizhnik equation (1.1) in closed form in terms of elementary, Lambert and hypergeometric functions as well as in parametric or implicit form. The main tool for this purpose was the optimized procedure of Lie reduction. A rigorous description and a proper substantiation of this procedure is in fact the main theoretical attainment of the second chapter.

Using the results of Chapter 1 on the maximal Lie invariance algebras $\mathfrak{g}$ and $\mathfrak{g}_{\rm L}$ of the equation (1.1) and of its nonlinear Lax representation (1.14) and their point-symmetry pseudogroups G and $G_{\rm L}$, see also [39], we have classified one- and two-dimensional subalgebras of the algebra $\mathfrak{g}$ and one-dimensional subalgebras of the algebra $\mathfrak{g}_{\rm L}$ up to the G- and $G_{\rm L}$-equivalences, respectively. We could only classify subalgebras that are appropriate for Lie reduction but this would not result in an essential simplification in comparison with the classification of all one- and two-dimensional subalgebras and the further selection of the appropriate ones among the listed inequivalent subalgebras. Instead of the standard equivalences within the algebras $\mathfrak{g}$ and $\mathfrak{g}_{\rm L}$ up to their inner automorphisms, which coincide with the $G_{\rm id}$- and $G_{\rm L,id}$-equivalences, where $G_{\rm id}$- and $G_{\rm L,id}$ are the identity components of G and $G_{\rm L}$, respectively, we have used the stronger G- and $G_{\rm L}$-equivalences. In this way, we have also taken into account the discrete point symmetry transformations of the equation (1.1), which has allowed us to reduce the optimal lists of subalgebras. Moreover, as explained in Section 2.1, it has also made the Lie reduction procedure consistent with the natural G-equivalence on the solution set of the equation (1.1). The above arguments clearly confirm that the correct computation of G and G_{L} in Chapter 1 was important. Note that in fact the algebras $\mathfrak{g}$ and $\mathfrak{g}_{\mathrm{L}}$ are infinite-dimensional Lie pseudoalgebras of vector fields. In general, the classification of (low-dimensional) subalgebras of such an algebra is complicated, in particular, by the necessity of considering differential [54] or even functional [44, 88] equations in the course of this classification.

The algebra $\mathfrak{g}$ is injectively mapped into the algebra $\mathfrak{g}_{\mathrm{L}}$ via extending the vector fields from $\mathfrak{g}$ to the variable v. The vector fields (1.2), which span $\mathfrak{g}$, are extended trivially and formally coincide with their counterparts in $\mathfrak{g}_{\mathrm{L}}$. The only exception is the vector field D^{s}, which extends to $\bar{D}^{\mathrm{s}}=x\partial_x+y\partial_y+3u\partial_u+\frac{3}{2}v\partial_v$. Moreover, $\mathfrak{g}_{\mathrm{L}}=\bar{\mathfrak{g}}\in\langle\bar{P}^v\rangle$, where $\bar{\mathfrak{g}}$ is the image of $\mathfrak{g}$ under the above mapping, and $\bar{P}^v=\partial_v$. Although the corresponding homomorphism[2.5] of the pseudogroup G into the pseudogroup G_{L} is not injective, its kernel is generated by the discrete involution $\mathcal{J}^{\mathrm{s}}:=\mathcal{D}^{\mathrm{s}}(-1)$ from G, which of course involves the restrictions of $\mathcal{J}^{\mathrm{s}}$ as well, and the quotient pseudogroups $G/\{\mathrm{id},\mathcal{J}^{\mathrm{s}}\}$ and $G_{\mathrm{L}}/\{\bar{\mathcal{P}}^v(B),\bar{\mathcal{J}}^v\circ\bar{\mathcal{P}}^v(B)\mid B\in\mathbb{R}\}$ are isomorphic, see the paragraph after Theorem 1.13. As a result, the classifications of one- and two-dimensional subalgebras of the algebra $\mathfrak{g}_{\mathrm{L}}$ up to the G_{L}-equivalence can be easily derived from the respective classifications for the algebra $\mathfrak{g}$ up to the G-equivalence, cf. Lemmas 2.3 and 2.7 for the case of dimension one. Nevertheless, we have not presented the classification of two-dimensional subalgebras of the algebra $\mathfrak{g}_{\mathrm{L}}$ since we needed only a few of these subalgebras, which are given directly when using them for Lie reductions of the nonlinear Lax representation (1.14) in Section 2.5.2. The correspondence between the equivalence classes of one-dimensional (resp. two-dimensional) subalgebras of the algebras $\mathfrak{g}$ and $\mathfrak{g}_{\mathrm{L}}$ is injective but not one-to-one. The list of inequivalent subalgebras of $\mathfrak{g}$ of any fixed dimension can be trivially embedded in the corresponding list for the algebra $\mathfrak{g}_{\mathrm{L}}$

[2.5] For this homomorphism and the isomorphism below, we should replace G by its trivial prolongation to v, considering the restriction of elements of the prolongation on open subsets of the space with the coordinates (t,x,u,v).

via the above extension of elements of $\mathfrak g$ to the variable v. The bijection breaking is related to the disappearance of $\mathcal J^{\rm s}$ and the appearance of $\bar P^v$ in the course of the transition from $(G,\mathfrak g)$ to $(G_{\rm L},\mathfrak g_{\rm L})$, see the subalgebras $\bar{\mathfrak s}^0_{2.1'}$, $\bar{\mathfrak s}^{2/3}_{2.1'}$ and $\bar{\mathfrak s}^{\varepsilon\nu}_{2.4}$ in Section 2.5.2. The last family of subalgebras is the most interesting since, in contrast to the corresponding coefficient in the second basis vector field of $\tilde{\mathfrak s}_{2.4}$ and the G-equivalence, the parameter ε in $\bar{\mathfrak s}^{\varepsilon\nu}_{2.4}$ cannot be set to 1 up to the $G_{\rm L}$-equivalence.

The described relation between the lists of inequivalent subalgebras of $\mathfrak g$ and of $\mathfrak g_{\rm L}$ can be reformulated in terms of the relation between the corresponding collections of inequivalent Lie reductions of the equation (1.1) and of the nonlinear Lax representation (1.14).

If subalgebras of $\mathfrak g$ are G-equivalent, then the corresponding reduced equations are necessarily similar with respect to point transformations of the invariant variables. Consider a class $\mathcal C$ of reduced equations for the equation (1.1) that is associated with a parameterized family $\mathcal F$ of subalgebras of $\mathfrak g$, and thus the arbitrary elements of $\mathcal C$ are expressed in terms of the subalgebra parameters. Then the stabilizer of $\mathcal F$ in G induces a (pseudo)subgroup $G^\sim_{\mathcal C,\rm ind}$ of the equivalence (pseudo)group $G^\sim_{\mathcal C}$ of the class $\mathcal C$. The proper inclusion $G^\sim_{\mathcal C,\rm ind}\varsubsetneq G^\sim_{\mathcal C}$, which happens quite commonly, means that some elements of $G^\sim_{\mathcal C}$ are not induced by transformations from G, and hence we call them *hidden equivalence transformations* of the class $\mathcal C$. If the subalgebras from the family $\mathcal F$ are G-inequivalent to each other, then transformations from the group G can induce only point symmetries of equations from the class $\mathcal C$ but not point transformations between different elements of this class. At the same time, in the case of the presence of hidden equivalence transformations, a wide subset of the action groupoid of $G^\sim_{\mathcal C}$ can still be used for mapping the class $\mathcal C$ to its proper subclass $\mathcal C'$, which formally has less number of (significant) arbitrary elements.[2.6] Then the correspondence between the parameters of

[2.6] See [102] and references therein for mappings between classes of differential equations that are generated by families of point transformations.

the family $\mathcal{F}$ and the arbitrary elements of the subclass $\mathcal{C}'$ is definitely not injective. We can select ansatzes associated with subalgebras of $\mathcal{F}$ such that the corresponding class of reduced equations is minimal up to the described mappings by hidden equivalence transformations. Nevertheless, this is not always convenient as shown by reductions 1.3 and 1.4. The classes of reduced equations 2.5, 2.9 and 2.14 are also not minimal in the above sense. In this context, the family $\mathcal{F}$ of subalgebras $\{\mathfrak{s}^{\rho}_{1.3}\}$ is especially demonstrative. After excluding the singular subalgebra $\mathfrak{s}^{1}_{1.3}$ from $\mathcal{F}$ and properly modifying the corresponding ansatzes from Table 2.1, we have derived the single simple reduced equation (2.5) instead of a class of reduced equations with the functional parameter $\rho = \rho(t)$ of $\mathcal{F}$ as its arbitrary element.

We have paid considerable attention to the selection of optimal ansatzes and thus simplified the further consideration but the simplification is not as significant as, e.g., that achieved for the Navier–Stokes equations in [53, 54, 118]. Most of the reduced equations for the equation (1.1) are quite cumbersome, and this is not the only feature of them that complicates the computation of their Lie and discrete point symmetries. Thus, each of reduced equations 1.1^0, 2.1^κ (including 2.13), 2.2^ν and $2.14^{0\nu\delta'}$ is not of maximal rank on the entire associated manifold in the corresponding jet space.[2.7] Even if a reduced equation is of maximal rank, it is not necessarily can be represented in the normal form, see reduced equation $2.5^{\lambda\mu}$ with $\lambda = 2/3$. As far as we know, Lie and general point symmetries of such unusual differential equations have not been considered in the literature. For some reductions of codimension two, even under the optimal choice of ansatzes, the permutation $\mathfrak{J}$ of x and y, which is a simple and obvious discrete point symmetry of the equation (1.1), induces more complicated and nontrivial discrete point symmetries of the corresponding reduced equations, and this leads to the complexity of general elements of the point

[2.7] Each of the reductions 2.1^κ (including 2.13), 2.2^ν and $2.14^{0\nu\delta'}$ can be considered as a two-step Lie reduction with reduction 1.1^0 as its first step.

symmetry groups of some reduced ordinary differential equations. There is no similar phenomenon for Lie symmetries of reduced equations obtained from the equation (1.1). Nevertheless, we have comprehensively studied point symmetries and their induction for all the reduced equations selected in the course of applying the optimized Lie reduction procedure to the equation (1.1). This study itself is a necessary ingredient of the reduction procedure. It has helped us to cut down the number of Lie reductions to be considered and to integrate or at least to lower the order of reduced ordinary differential equations. Note that discrete symmetries of reduced equations have been computed for the first time in [127], which is the source of this chapter. In view of the above reasons such as the complexity of reduced equations and their point symmetries and the simplicity of their Lie-symmetry vector fields, the algebraic method by Hydon and its various modifications are especially efficient and convenient for this computation.

For finding exact solutions of reduced ordinary differential equations, we have also used the associated reduced systems for the nonlinear Lax representation (1.14). Due to properly arranging the hierarchy of Lie reductions of the equation (1.1) and accurately selecting a low number of reduced ordinary differential equations to be integrated, we were able to deeply analyze them and construct wider families of exact solutions of the equation (1.1) than those presented in the literature. Of course, there are a number of possibilities for extending and generalizing the results of this chapter. In particular, since most of Lie symmetries of the reduced equation (2.6) are hidden for the original equation (1.1), one can actually represent more solutions from the family (2.5) in an explicit form by means of Lie reductions of (2.6) than those found in Section 2.5.1. In addition, the results of Section 2.7 on multiplicative separation of variables for the equation (1.1) and of [92] on solutions of (1.1) that are polynomial in (x, y) show that more closed-form solutions of (1.1) can be constructed using other tools of symmetry analysis of differential equations.

We have checked all the obtained solutions and have selected only G-inequivalent ones among them. Checking constructed solutions and their inequivalence to known solutions is the last but most important step of any procedure of finding exact solutions of differential equations. Unfortunately, this step is commonly disregarded, which led to many papers containing only incorrect or known solutions, see the discussion of such papers in [74, 117].

Conclusion

In the thesis, we carried out the extended symmetry analysis of the (real symmetric potential) dispersionless Nizhnik equation. In the course of this study, we observed a number of new interesting phenomena and formalized, enhanced and developed several methods and techniques of group analysis of differential equations. In particular, we obtained the following results:

- Applying an original megaideal-based version of the algebraic method, we computed the point-symmetry pseudogroups G, $G_{\rm L}$ and $G_{\rm dN}$ of the dispersionless Nizhnik equation, the corresponding nonlinear Lax representation and the dispersionless counterpart of the symmetric Nizhnik system as well as the contact-symmetry pseudogroup $G_{\rm c}$ of this equation, which is the first usage of the megaideal-based version of the algebraic method for finding the contact-symmetry (pseudo)group of a differential equation. It turned out the pseudogroup $G_{\rm c}$ coincides with the first prolongation of the pseudogroup G.
- As the first step of the above computations, we studied the structure of the maximal Lie invariance algebras of the systems of differential equations under consideration and constructed sufficient sets of megaideals of these algebras, main of which are their radicals. For constructing a megaideal of the maximal Lie invariance algebra of the nonlinear Lax representation of the dispersionless Nizhnik equation, we developed a new technique, which is completely different from existing techniques that are used for the same purpose.
- It was shown that the necessary algebraic condition completely defines the point-symmetry pseudogroup of the dispersionless Nizhnik

equation. This gave the first example of a system of differential equations with this property in the literature. Even the nonlinear Lax representation of the dispersionless Nizhnik equation and the dispersionless counterpart of the symmetric Nizhnik system do not have this property.

- We checked whether the subalgebras of the maximal Lie invariance algebra $\mathfrak{g}$ of the dispersionless Nizhnik equation that naturally arise in the course of the computation of the pseudogroup G define the diffeomorphisms stabilizing this algebra or its first prolongation.

- We constructed all the third-order partial differential equations in three independent variables that are invariant with respect to the algebra $\mathfrak{g}$. We found a set of geometric properties of the dispersionless Nizhnik equation that exhaustively defines it. In addition to the invariance with respect to the algebra $\mathfrak{g}$, it includes the presence of the three simplest conservation-law characteristics 1, u_{xx} and u_{yy}. This combines an inverse group classification problem with an inverse problem on conservation laws.

- The one- and two-dimensional subalgebras of the algebra $\mathfrak{g}$ are exhaustively classified up to the equivalence generated by the pseudogroup G, which led to the complete classification of Lie reductions of the dispersionless Nizhnik equation to partial differential equations with two independent variables and to ordinary differential equations. We also showed that Lie reductions of this equation to algebraic equations give no its new solutions as compared to those constructed using Lie reductions of codimensions two and three.

- Lie and point symmetries of the derived reduced equations are comprehensively studied, including the analysis of which of them correspond to hidden symmetries of the original equation. The point symmetry groups of reduced equations, in particular those that are not of maximal rank, were computed for the first time, including their discrete point symmetries. It turned out that in contrast to Lie symmetries, simple and obvious discrete point symmetries of the initial equation, even under the optimal choice of ansatzes, can induce complicated and nontrivial discrete point symmetries of the corresponding reduced equations.

- The wide families of new exact invariant solutions of the dispersionless Nizhnik equation are constructed in closed form in terms of elementary, Lambert and hypergeometric functions as well as in parametric or implicit form.
- Multiplicative separation of variables was used for illustrative construction of families of non-invariant solutions of the dispersionless Nizhnik equation, which essentially generalizes some obtained families of invariant solutions of this equation.

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Appendix A

List of publications and approbation of results

This appendix contains the list of publications of the PhD candidate on the thesis' topic as well as information about the approbation of the thesis' results.

Scientific works in which the scientific results of the thesis were published:

1. Boyko V.M., Popovych R.O. and Vinnichenko O.O., Point- and contact-symmetry pseudogroups of dispersionless Nizhnik equation, *Commun. Nonlinear Sci. Numer. Simul.* **132** (2024), 107915, 19 pp., doi:10.1016/j.cnsns.2024.107915, arXiv:2211.09759. (Scopus – Q1, WoS – Q1, SJR – Q1).

2. Vinnichenko O.O., Boyko V.M. and Popovych R.O., Lie reductions and exact solutions of dispersionless Nizhnik equation, *Anal. Math. Phys.* **14** (2024), 82, 56 pp., doi:10.1007/s13324-024-00925-y, arXiv:2308.03744. (Scopus – Q1, WoS – Q2, SJR – Q1).

Scientific works certifying the approbation of the thesis' materials:

1. Вінніченко О.О., Бойко В.М., Попович Р.О., Псевдогрупи точкових і контактних симетрій бездисперсійного рівняння Нижника, Тези доповідей Міжнародного симпозіуму "Симетрія та інтегровність рівнянь математичної фізики", Київ, Інститут математики НАН Укра-

їни, 2022, https://www.imath.kiev.ua/~appmath/Abstracts2022/Vinnichenko.html.

2. Вінніченко О.О., Псевдогрупи точкових і контактних симетрій бездисперсійного рівняння Нижника та його нелінійного представлення Лакса, Тези доповідей Міжнародної конференції молодих математиків, Київ, Інститут математики НАН України, 2023, https://www.imath.kiev.ua/~young/youngconf2023/Abstracts_2023/DEMF/Vinnichenko.pdf.

3. Вінніченко О.О., Класичний симетрійний аналіз бездисперсійного рівняння Нижника, Abstracts of the Workshop Complex Dynamical Systems in the Science: Theory, Mathematical Modelling, Computing and Application, Київ, Інститут математики НАН України, 2023, https://drive.google.com/file/d/1DUpp8UXjrM6rlOL-xdU42l-3A-diho0y.

4. Вінніченко О.О., Симетрійні розв'язки бездисперсійного рівняння Нижника, Тези доповідей XII Всеукраїнської наукової конференції молодих математиків, Київ, Національний університет "Києво-Могилянська академія", 2024, https://mathconf.ukma.edu.ua/mathconf_xii.pdf.

5. Вінніченко О.О., Мультиплікативне розділення змінних для знаходження розв'язків бездисперсійного рівняння Нижника, Тези доповідей Конференції молодих учених "Підстригачівські читання – 2024", Інститут прикладних проблем механіки і математики ім. Я.С. Підстригача НАН України, Львів, 2024, http://iapmm.lviv.ua/chyt2024/abstracts/Vinnichenko.pdf.

6. Vinnichenko O.O., Boyko V.M. and Popovych R.O., Geometric and algebraic properties of dispersionless Nizhnik equation, Abstracts of the International Scientific Online Conference "Algebraic and Geo-

metric Methods of Analysis”, Odesa National University of Technology, Odesa, 2024, https://imath.kiev.ua/~topology/conf/agma2024/abstracts/texts/vinnichenko/vinnichenko.pdf.

7. Vinnichenko O., Boyko V. and Popovych R., Geometric properties and exact solutions of dispersionless Nizhnik equation, Online materials of Bogolyubov Kyiv Conference “Problems of Theoretical and Mathematical Physics”, Kyiv, Ukraine, September 24–26, 2024, Abstract ID: 86, https://www.imath.kiev.ua/~institute/BKC2024/abstracts/Vinnichenko.pdf.

Information on the approbation of the thesis’ results. The main results of the thesis were reported and discussed at:

- Seminar of Department of Mathematical Physics of Institute of Mathematics of National Academy of Sciences of Ukraine (Kyiv, 2022–2024);
- International Symposium “Symmetry and Integrability of Equations of Mathematical Physics” (Kyiv, Institute of Mathematics of NAS of Ukraine, 2022);
- International Conference of Young Mathematicians (Kyiv, Institute of Mathematics of NAS of Ukraine, 2023);
- Workshop CDSS (Complex Dynamical Systems in the Science): theory, mathematical modelling, computing and application (Kyiv, Institute of Mathematics of NAS of Ukraine, 2023);
- Seminar of Young Scientists (Kyiv, Institute of Mathematics of NAS of Ukraine, 2024);
- XII All-Ukrainian Scientific Conference of Young Mathematicians (Kyiv, National University of Kyiv Mohyla Academy, 2024);

- Conference of Young Mathematicians "Pidstryhach readings – 2024" (Lviv, Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of NAS of Ukraine, 2024);
- International Scientific Online Conference "Algebraic and Geometric Methods of Analysis" (Odesa, Odesa National University of Technology, 2024);
- Bogolyubov Kyiv Conference "Problems of Theoretical and Mathematical Physics" (Kyiv, Institute of Mathematics of NAS of Ukraine, 2024).