On the generalized Skorokhod map for piecing together two processes

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A. Skorokhod in 1961 formulated the reflection problem in his work [1], which subsequently proved to be an effective method for constructing reflected diffusions. Based on this reflection problem, a mapping was built that assigns to a function the solution of Skorokhod's reflection problem for that function. This mapping turned out to be useful for solving various problems in queueing theory (see, for example, [2, 3, 4]). Skorokhod's reflection problem has been generalized by different authors. In particular, A. Pilipenko in the article [5] posed the problem of jump-like reflection, and this work made it possible to construct Brownian motion with jump-like reflection (see articles [6, 7]). In the article [8], A. Pilipenko and A. Sarantsev considered a system with switches and, as a result of certain limit transitions, were able to obtain in the limit the classical Skorokhod reflection, reflection with delay, absorbing reflection, and reflection with jump-like exit from the boundary.

Let f and g be two continuous functions. We aim to construct a dynamics whose increments match those of f in the upper half-plane and those of g in the lower half-plane. To achieve this, we first introduce a level $\delta > 0$ and consider dynamics with switching at certain levels. Specifically, we start with the dynamics of f at $t = 0$ and switch from the dynamics of f to g when $f(t)$ reaches the interval $(-\infty, -\delta]$. Then, we switch back from the dynamics of g to f when reaching the interval $[0, \infty)$. Then, we switch again to the dynamics of g when reaching the interval $(-\infty, -\delta]$, and so on. The function constructed according to these rules will be denoted by $\mathcal{G}_\delta(f, g)$. Our goal is to let δ approach zero and find the limit of the switching dynamics of $\mathcal{G}_\delta(f, g)$.

We introduce the notation:

$$m_f(t) := -\min_{s\in[0,t]}(f(s) \wedge 0), \qquad m_g(t) := \max_{s\in[0,t]}(g(s) \vee 0).$$

We call a number C a *level of constancy* of a function F, if there exist $t_1 < t_2$ such that $F(t) = C, \quad t \in [t_1, t_2]$.

Theorem. *If the functions m_f and m_g do not share any common level of constancy, then there exists a continuous function $\mathcal{G}$ such that $\mathcal{G}_\delta(f, g) \underset{\delta\to 0}{\longrightarrow} \mathcal{G}$ locally uniformly. This function $\mathcal{G}$ is given by the formula*

$$\mathcal{G}(t) = f(T_F(t)) + g(T_G(t)), \quad t \geq 0,$$

where the pair of non-negative functions T_F, T_G is the unique solution to the system of functional equations

$$\begin{cases} m_f(T_F(t)) = m_g(T_G(t)) \\ T_F(t) + T_G(t) = t, \ t \geq 0. \end{cases}$$

Remark. *Denote by $\hat{f} := f + m_f$ the $\hat{g} := g - m_g$ the Skorokhod reflection of f and g into positive and negative halflines, respectively. Then the limit function $\mathcal{G}$ has a representation*

$$\mathcal{G}(t) = \hat{f}(T_F(t)) + \hat{g}(T_G(t)), \quad t \geq 0.$$

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