On Hausdorff dimension of Julia sets

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Introduction

Discrete dynamical system associated to a polynomial

Let $f(z) = a_0 z^d + a_1 z^{d-1} + \ldots + a_{d-1} z + a_d$ be a polynomial, $d \in \mathbb{N}, a_i \in \mathbb{C}, z \in \mathbb{C}$. For $n \in \mathbb{N}$ denote by $f^n(z) = f(f(\ldots(f(z))\ldots))$ (iterated n times).

Discrete dynamical system associated to a function $f(z)$: given z_0 consider its orbit
$O(z_0) = \{z_0, z_1 = f(z_0), z_2 = f(z_1), z_3 = f(z_2), \ldots\}$, $z_n = f^n(z)$.

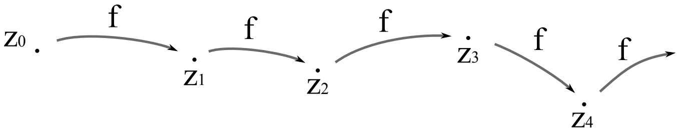


Figure: Orbit of a point.

The Julia set: informal definition

An important question is how orbits depend on the initial parameter z_0.

Natural dichotomy:
• regular behavior (slight change in the initial condition does not affect much the long-time behavior which can be therefore accurately predicted);
• chaotic behavior (arbitrary small variation of the initial condition may change unpredictably the long-time behavior).

Informally speaking, for a polynomial (or rational) function $f(z)$ the set of parameters z_0 producing chaotic behavior is called the *Julia set* J_f. Regular parameters constitute the Fatou set F_f.

The Julia set: formal definition

Definition

A family $\mathcal{F}$ of complex analytic functions is called normal if any sequence $\{f_i\} \subset \mathcal{F}$ has a convergent subsequence.

Definition

The Fatou set F_f of a polynomial f is the set of points $z \in \mathbb{C}$ having a neighborhood $U(z)$ such that the iterates $f^n|_{U(z)}$ form a normal family. The Julia sets is the complement of the Fatou set J_f.

Filled Julia set $K_f = \{z \in \mathbb{C} : \{f^n(z)\}_{n \in \mathbb{N}} \text{ is bounded}\}$.
Julia set $J_f = \partial K_f$.

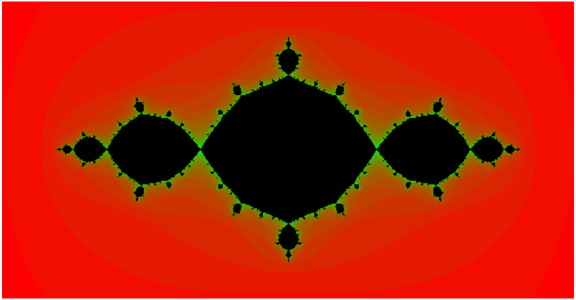


Figure: The Basilica Julia set, $f(z) = z^2 - 1$.

Julia set of a polynomial f

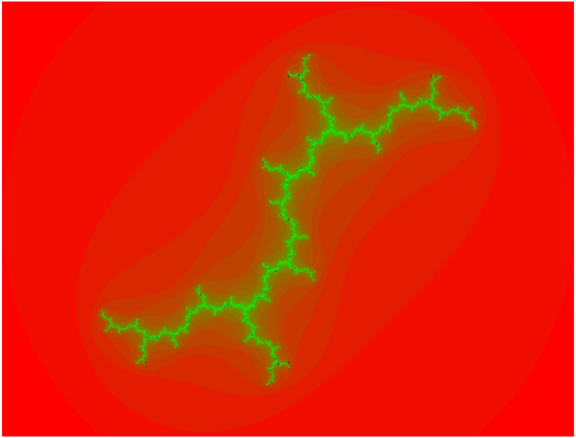


Figure: A dendrite Julia set, $f(z) = z^2 + i$.

Julia set of a polynomial f

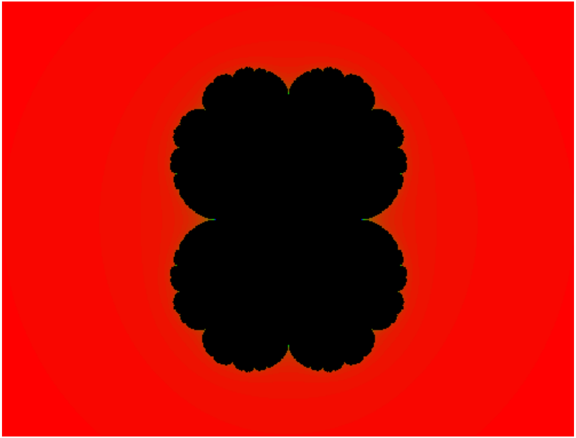


Figure: The cauliflower, $f(z) = z^2 + 0.25$.

Julia set of a polynomial f

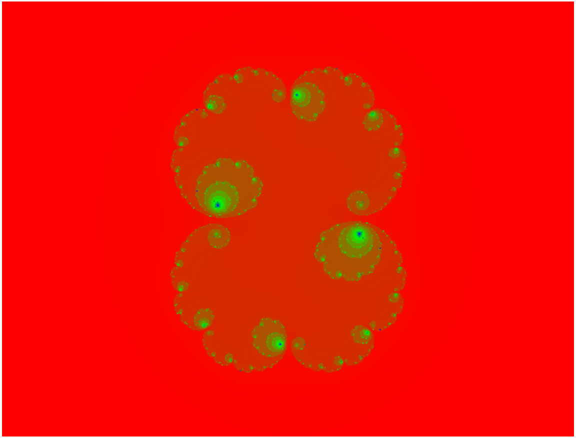


Figure: A perturbation of the cauliflower, $f(z) = z^2 + 0.26 + 0.001i$.

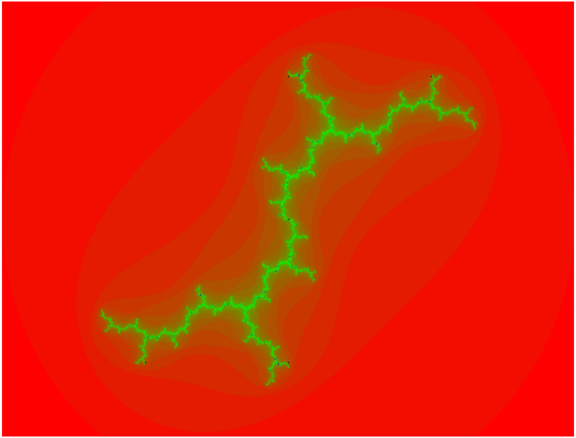
Properties of Julia sets

Complete invariance

Proposition

The Julia set of a polynomial $f(z)$ is completely invariant under the action of f: $f(J_f) = f^{-1}(J_f) = J_f$.

Recall: $K_f = \{z \in \mathbb{C} : \{f^n(z)\}_{n \in \mathbb{N}} \text{ is bounded}\}, \; J_f = \partial K_f$.



Self-similarity

Definition

A family $\mathcal{F}$ of complex analytic functions is called normal if any sequence $\{f_i\} \subset \mathcal{F}$ has a convergent subsequence.

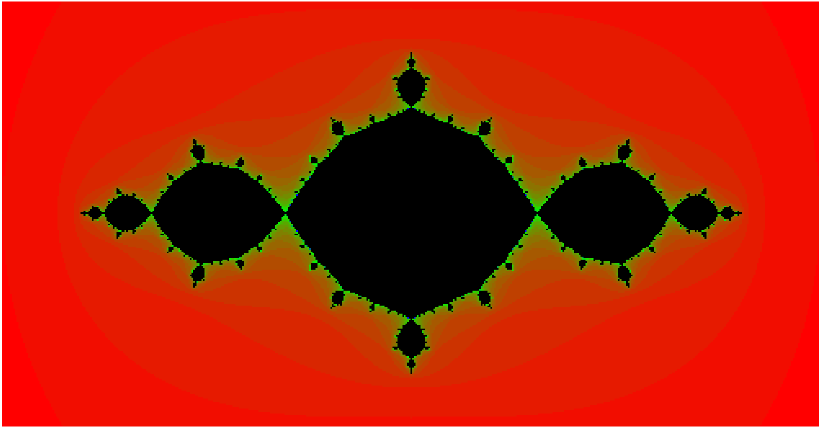
Theorem (Montel's Theorem)

Any family of complex analytic functions, all of which omit the same two values $a, b \in \mathbb{C}$, is normal.

Corollary

Let $f(z)$ be a polynomial. For any $z \in J_f$ and any neighborhood $U(z)$ of z the union $\cup_{n \in \mathbb{N}} f^n(U(z))$ covers whole complex plane, except possibly one exceptional value.

Self-similarity



Critical points

A point $a \in \mathbb{C}$ such that $f'(a) = 0$ is called a *critical point* of f. Near a critical point f is not one-to-one.

Consider $p_c(z) = z^2 + c$. Take a large disk $U_R(0)$ and start taking preimages under p_c.

Case a) $p_c^{-n}(U_R(0))$ contains the critical value $c = p_c(0)$ for all $n \in \mathbb{N}$. Then $K(p_c) = \cap p_c^{-n}(U_R(0))$ is connected and $J(p_c) = \partial K(p_c)$ is connected.

b) $c \notin p_c^{-n}(U_R(0))$ for some $n \in \mathbb{N}$. Then $K(p_c) = J(p_c)$ is totally disconnected.

A connected Julia set

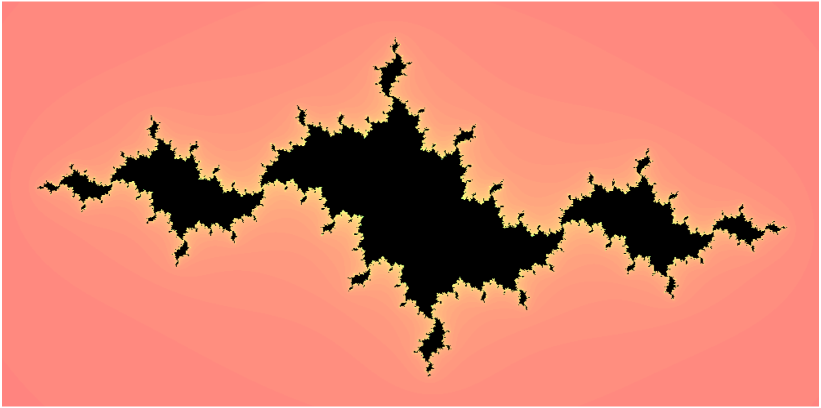


Figure: The Julia set of $f(z) = z^2 - 1.1 + 0.2i$.

A totally disconnected Julia set

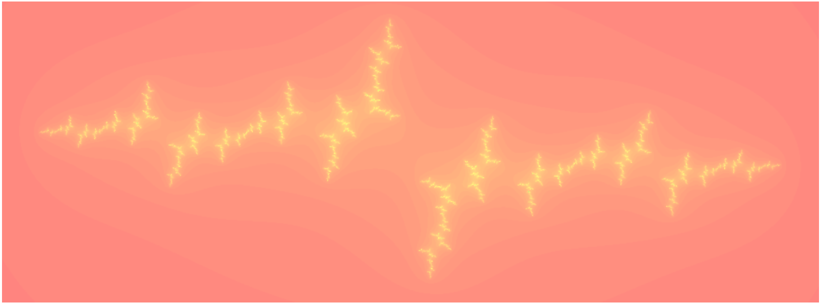
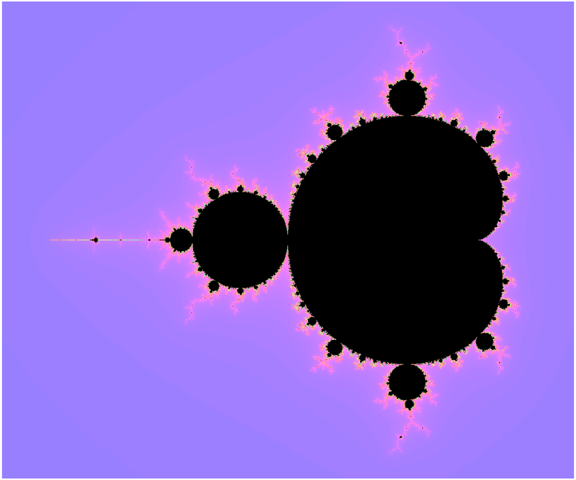


Figure: The Julia set of $f(z) = z^2 - 1.4 - 0.2i$.

Mandelbrot set

The Mandelbrot set $\mathcal{M}$ is the set of parameters $c \in \mathbb{C}$ such that $\{p_c^n(0)\}_{n\in\mathbb{N}}$ is bounded (equivalently, $K(p_c)$ and $J(p_c)$ are connected).



Periodic points

A point $a \in \mathbb{C}$ is called *periodic with period n* for a polynomial $f(z)$ is $f^n(a) = a$. The *multiplier* of a is $\lambda = Df^n(a)$. The periodic point a is called

- *attracting*, if $|\lambda| < 1$,
- *repelling*, if $|\lambda| > 1$,
- *indifferent*, if $|\lambda| = 1$.

A point $c \in \mathbb{C}$ is called a *critical point* of $f(z)$ if $f'(z) = 0$.

Theorem (Fatou,Shishikura)

A polynomial of degree d has at most $d-1$ non-repelling periodic orbits. Moreover, each non-repelling periodic orbit has a critical point of $f(z)$ associated to it.

Attracting periodic orbits

For any attracting periodic point a of period n of a polynomial $f(z)$ there exists a neighborhood $U(a)$ such that $f^{kn}(z) \to a$ uniformly on $U(a)$. Moreover, on $U(a)$ the map f^n is conformally conjugate to multiplication by λ: there exists a biholomorphic map $\phi : U(a) \to \mathbb{C}$ such that

$$\phi \circ f^n \circ \phi^{-1}(z) = \lambda z \ \ \forall \ \ z \in \phi(U(a)).$$

The set of points $z \in \mathbb{C}$ whose orbit is attracted to the orbit of $\mathcal{O}$ of a is called the *attracting basin* $B(\mathcal{O})$ of the attracting orbit. One has: $B(\mathcal{O}) \subset F(f)$, $\partial B(\mathcal{O}) \subset J(f)$.

Attracting periodic orbits

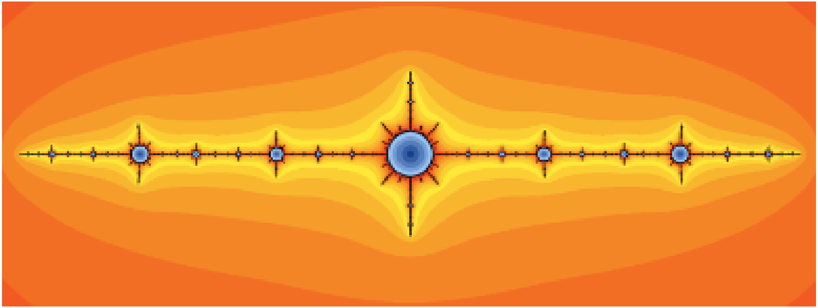
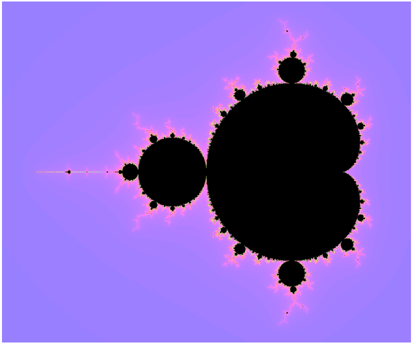


Figure: The airplane Julia set, $f(z) = z^2 - 1.754877...$

Hyperbolicity

For polynomials (or rational functions) having only attracting and repelling periodic orbits their Julia sets are called *hyperbolic*. Such a polynomial $f(z)$ is *strictly expanding* on its Julia set: there exists $\alpha > 1, n \in \mathbb{N}$ such that $|Df^n(z)| \geqslant \alpha$ for all $z \in J(f)$.

Density of Hyperbolicity Conjecture: parameters $c \in \mathbb{C}$ such that $J(p_c)$ is hyperbolic are dense in $\mathcal{M}$.



Repelling periodic orbits

Recall, a is a period n repelling periodic of $f(z)$ if $f^n(z) = z$ and $|\lambda| = |Df^n(a)| > 1$. Since $|Df^k(a)| \to \infty$ we have that $\{f^k\}_{k\in\mathbb{N}}$ is not a normal family, so $a \in J(f)$.

Theorem

Repelling periodic points are dense in $J(f)$.

Idea of the proof: use

Montel's Theorem. Any family of complex analytic functions, all of which omit the same two values $a, b \in \mathbb{C}$, is normal.

assume that $\zeta \in J(f)$, $U(\zeta)$ is a neighborhood of ζ and $f^k(z) \neq z$ for all $k \in \mathbb{N}$. Then we have a family of nonzero functions $p_k(z) = f^k(z) - z$ on $U(\zeta)$. If we knew that for some $a \neq 0$ we have $p_k(z) \neq a$ for all $k \in \mathbb{N}$ we could use Montel's Theorem to show $\{p_k\}_{k\in\mathbb{N}}$ is normal, and $\zeta \in F(f)$. One can do this with slightly more complicated p_k.

Neutral periodic points

For a polynomial $f(z)$ a periodic point a of period n is called *neutral* if $|Df^n(a)| = 1$. Let $Df^n(a) = \exp(2\pi i\theta)$. There are three types of neutral fixed points:

1) parabolic, for $\theta \in \mathbb{Q}$.

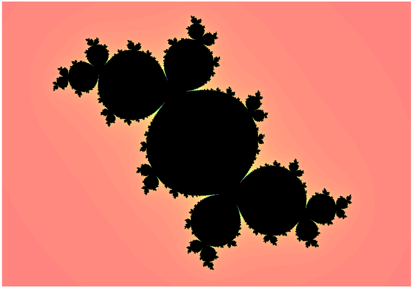


Figure: $K(p_c)$ with $c \approx -0.125 + 0.64952i$, $n = 1$, $\theta = 2\pi i/3$.

Neutral periodic points

For a polynomial $f(z)$ a periodic point a of period n is called *neutral* if $|Df^n(a)| = 1$. Let $Df^n(a) = \exp(2\pi i\theta)$. There are three types of neutral fixed points:

2) Siegel, $\theta \in \mathbb{R} \setminus \mathbb{Q}$, f^n is conjugated to a rotation near a.

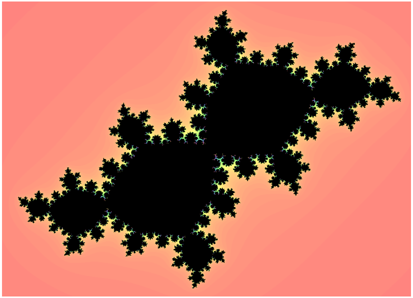


Figure: $K(p_c)$ with $c \approx -0.39054 - 0.58679i$, $n = 1$, $\theta = 2\pi\frac{\sqrt{5}-1}{2}i$.

Neutral periodic points

For a polynomial $f(z)$ a periodic point a of period n is called *neutral* if $|Df^k(a)| = 1$. Let $Df^k(a) = \exp(2\pi i\theta)$. There are three types of neutral fixed points:
3) Cremer, the remaining case. No plausible image of a Cremer Julia sets is known.

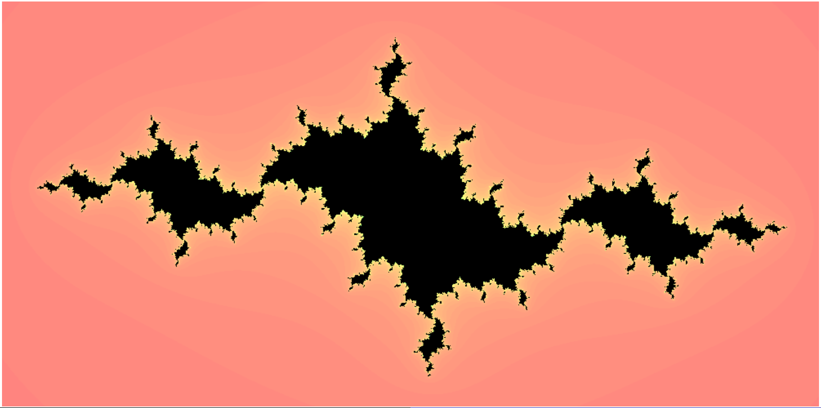
For a quadratic map $p_c(z) = z^2 + c$ with a Cremer periodic point:

- the Julia set J_c is not locally connected (Douady-Sullivan '83);
- there exist "hedgehogs" (fully invariant compact subsets) inside any neighborhood of the Cremer fixed point(Perez-Marco '97);
- there are non-trivial periodic cycles in arbitrarily small neighborhoods of the Cremer fixed point ("small cycle property", Yoccoz '87).

Classification of Fatou components

Theorem (Sullivan, '85)

For a polynomial $f(z)$ all connected components of the Fatou set are either periodic or preperiodic. Each such component either belongs to the attracting basin of an attracting or parabolic periodic point, or is mapped eventually onto a Siegel disk.



Area and HD of quadratic Julia sets

Notations: $f_c(z) = z^2 + c,\ \mathcal{J}_c = \mathcal{J}_{f_c}$.

- Ruelle '82: $\dim_{\mathrm{H}}(\mathcal{J}_c)$ is real-analytic in c on hyperbolic components and outside of the Mandelbrot set.
- Shishikura '98: for a generic $c \in \partial\mathcal{M}$ one has $\dim_{\mathrm{H}}(\mathcal{J}_c) = 2$.
- McMullen '98, Jenkinson-Pollicott '02: effective algorithms for computing $\dim_{\mathrm{H}}$ of attractors of conformal expanding dynamical systems (*e.g.* hyperbolic Julia sets).
- Avila-Lyubich '08: there exist Feiganbaum polynomials f_c with $\dim_{\mathrm{H}}(J_c) < 2$.
- Buff-Cheritat '12: there exist quadratic polynomials f_c with $\mathrm{area}(f_c) > 0$ *a*) having a Siegel fixed point, *b*) a Cremer fixed point, *c*) infinitely satellite renormalizable.
- Avila-Lyubich '22: there exist Feigenbaum polynomials f_c with $\mathrm{area}(\mathcal{J}_c) > 0$.

Estimates on HD from below using McMullen's matrices.

The Markov partition

Let J_c be the Julia set of a quadratic map $p_c(z) = z^2 + c$. Let $c \in \mathbb{R}$. We have $J_c \subset D_R(0) = \{z \in \mathbb{C} : |z| < R\}$ for some $R > 0$. Let $\{P_1^{(k)}, \ldots, P_{2^k}^{(k)}\}$ the set of connected components of $p_c^{1-k}(D_R(0) \setminus \mathbb{R})$.

Tiles for the Feigenbaum map $p_{\mathrm{Feig}}(z) = z^2 - 1.4011\ldots$

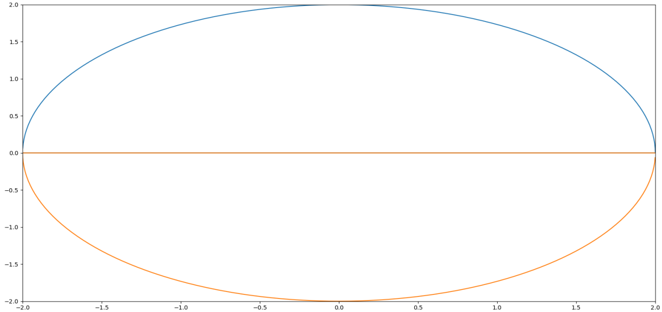


Figure: The partition of level 1.

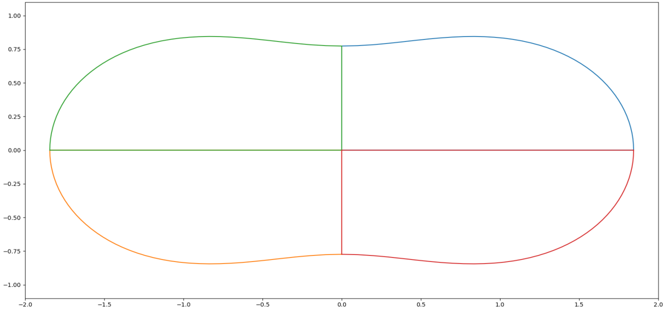


Figure: The partition of level 2.

Tiles for the Feigenbaum map $p_{\mathrm{Feig}}(z) = z^2 - 1.4011\ldots$

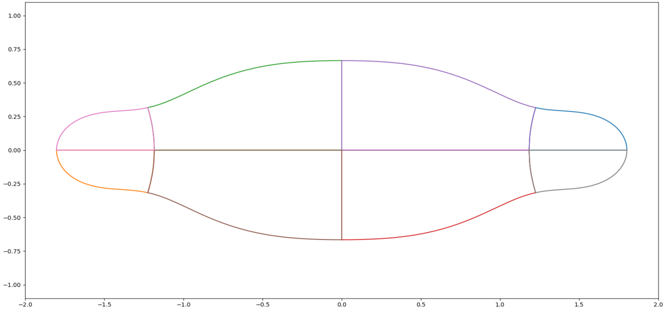


Figure: The partition of level 3.

Tiles for the Feigenbaum map $p_{\mathrm{Feig}}(z) = z^2 - 1.4011\ldots$

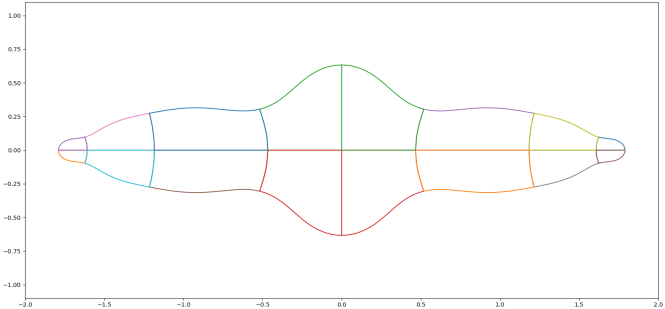


Figure: The partition of level 4.

Tiles for the Feigenbaum map $p_{\mathrm{Feig}}(z) = z^2 - 1.4011\ldots$

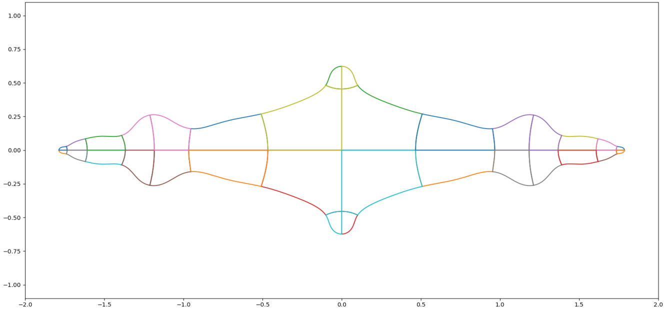


Figure: The partition of level 5.

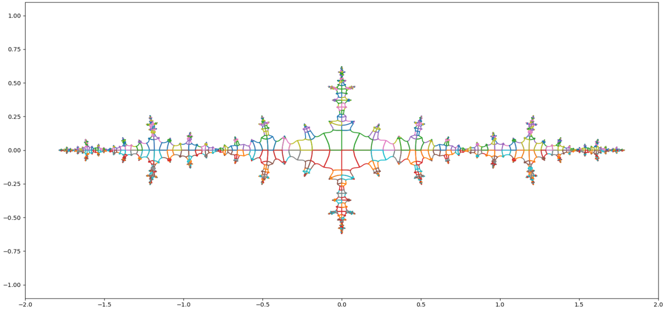


Figure: The partition of level 12.

The Markov Partition

Let J_c be the Julia set of a quadratic map $p_c(z) = z^2 + c$. Let $c \in \mathbb{R}$. We have $J_c \subset D_R(0) = \{z \in \mathbb{C} : |z| < R\}$ for some $R > 0$. Let $\{P_1^{(k)}, \dots, P_{2^k}^{(k)}\}$ the set of connected components of $p_c^{1-k}(D_R(0) \setminus \mathbb{R})$. Introduce the matrix $M^{(k)}$ with entries given by:

$$m_{i,j}^{(k)} = \left(\max\left\{|p_c'(z)| \colon z \in P_i^{(k)} \cap p_c^{-1}(P_j^{(k)})\right\}\right)^{-1} \tag{1}$$

if $P_i^{(k)} \cap p_c^{-1}(P_j^{(k)}) \neq \emptyset$, and $m_{i,j}^{(k)} = 0$ otherwise, $1 \leqslant i, j \leqslant 2^k$. Let $M^{(k)}(t)$ be the matrix with entries $(m_{i,j}^{(k)})^t$. Set

$$\delta_k = \inf\{t > 0 : \rho(M^{(k)}(t)) = 1\}. \tag{2}$$

Theorem (Real quadratic case)

Let $c \in \mathbb{R}$ and δ_k be as above. Then $\delta_k < \dim_{\mathrm{H}}(J_c)$ for $k \in \mathbb{N}$.

Applications

Application 1: the Feigenbaum quadratic polynomial

Recall that the Feigenbaum parameter $c_{\mathrm{Feig}} \approx -1.4011$ is the limit of the cascade of period-doubling parameters. Let $p_{\mathrm{Feig}} = z^2 + c_{\mathrm{Feig}}$.

Theorem (D.-Sutherland '20)

The Julia set of p_{Feig} has Hausdorff dimension less than two (and hence its Lebesgue measure is zero).

Theorem (D.-Gorbovickis-Tucker)

$\dim_{\mathrm{H}}(J_{p_{\mathrm{Feig}}}) > 1.4978.$

Application 2: smoothness of $\dim_{\mathrm{H}}(J_c)$ on $(c_{\mathrm{Feig}}, -3/4)$

Recall that a period m periodic point a of a holomorphic map $p(z)$ is called parabolic if $Dp^m(a) = \exp(2\pi\theta i)$ for some $\theta \in \mathbb{Q}$. Let $c_n \in (c_{\mathrm{Feig}}, 0)$ be the parameter such that p_{c_n} has period-2^n parabolic periodic point. It is known that the mapping $d(c) = \dim_{\mathrm{H}}(J_c)$ is smooth for c inside hyperbolic components of the Mandelbrot set (in particular, on (c_n, c_{n-1}) for any $n \in \mathbb{N}$). Jaksztas and Zinsmeister calculated asymptotics of the derivative $d'(c)$ for real c near c_n (depending on $d(c_n)$). Their results directly imply:

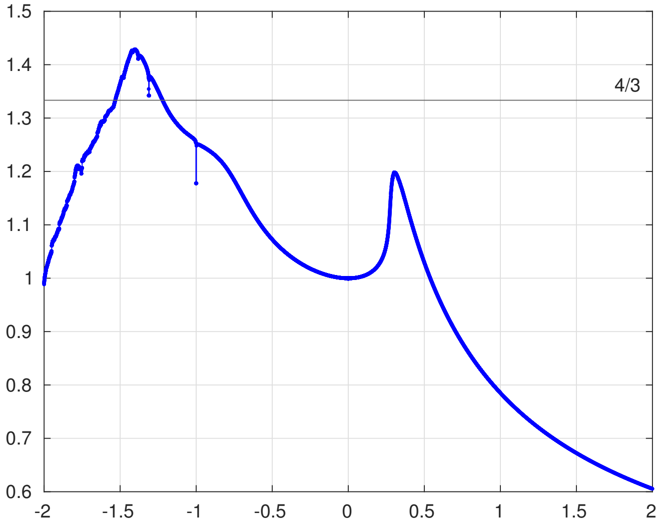
Theorem

For $n \in \mathbb{N}$ if $d(c_n) > 4/3$ then $d(c)$ is C^1-smooth near c_n.

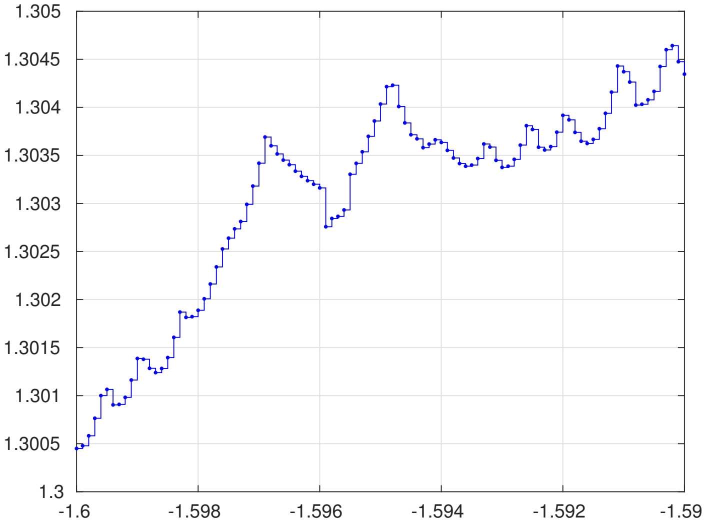
Theorem (D.-Gorbovickis-Tucker)

$\dim_{\mathrm{H}}(J_{p_{c_n}}) > 4/3$ *for all $n \in \mathbb{N}$, and so $d(c)$ is smooth on $(c_{\mathrm{Feig}}, -3/4)$.*

Application 3: graphs of the lower bound for $\dim_{\mathrm{H}}(J_c)$

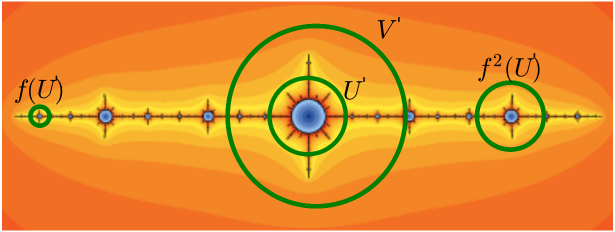


Application 3: graphs of the lower bound for $\dim_{\mathrm{H}}(J_c)$



Renormalization

A *quadratic-like map* is a ramified covering $f : U \to V$ of degree 2, where $U \Subset V$ are topological disks in $\mathbb{C}$.
A quadratic-like map f is called *renormalizable with period p* if there exist domains $U' \Subset U$ for which $f^p : U' \to V' = f^p(U')$ is a quadratic-like map.



The map $f^n|_{U'}$ is called a *pre-renormalization of f*; the map $\mathcal{R}_p f := \Lambda \circ f^p|_{U'} \circ \Lambda^{-1}$, where Λ is an appropriate rescaling of U', is the *renormalization of f*.

Stationary combinatorics

Assume that the domain U of definition of f is symmetric with respect to $\mathbb{R}$ and $0 \in U$ is the critical point of f. Denote a permutation $s = [s_0, \ldots, s_{n-1}]$ encoding the order of the points $0, f(0), \ldots, f^{n-1}(0)$ on the real line: $f^j(0) < f^k(0)$ for $0 \leqslant j, k \leqslant n-1$ if and only if $s_j < s_k$. A real infinitely renormalizable map f has stationary combinatorics if $s(\mathcal{R}^k f)$ does not depend on k.

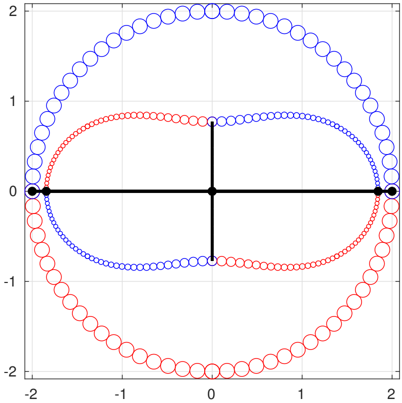
Application 4: stationary combinatorics

Permutation	Parameter c	Bound on $\dim_{\mathrm{H}}(J_c)$
$[1, 0, 5, 4, 3, 2]$	$-1.99638\ldots$	1.03142
$[1, 0, 4, 3, 2]$	$-1.98553\ldots$	1.07439
$[2, 0, 5, 4, 3, 1]$	$-1.96684\ldots$	1.08899
$[1, 0, 3, 2]$	$-1.94270\ldots$	1.15803
$[3, 0, 5, 4, 2, 1]$	$-1.90750\ldots$	1.14436
$[2, 0, 4, 3, 1]$	$-1.86222\ldots$	1.20002
$[1, 0, 2]$	$-1.78644\ldots$	1.29622
$[2, 0, 5, 3, 1, 4]$	$-1.78121\ldots$	1.32518
$[3, 0, 4, 2, 1]$	$-1.63192\ldots$	1.33306
$[4, 0, 5, 2, 3, 1]$	$-1.48318\ldots$	1.41584

Computational considerations

Approximating the tiles

To approximate rigorously the tiles of the Markov partition we cover their boundaries by disks.



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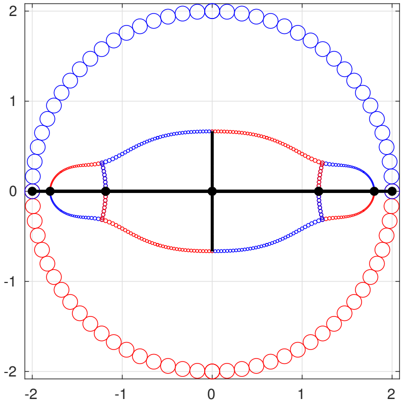

Circular complex arithmetic

To compute consecutive approximations we need to cover preimage of each disk under $z^2 + c$ by another disk. We represent disks by pairs

$$\mathbf{z} = (m, r) = \{z \in \mathbb{C} : |z - m|\}$$

and introduce operations on pairs. Set

$$\mathbf{z}_1 - \mathbf{z}_2 = (m_1 - m_2, r_1 + r_2).$$

Given $\mathbf{z} = (m, r)$ with $m = \rho e^{i\theta}$ assuming $0 \notin \mathbf{z}$ set

$$\alpha_1 = \sqrt{\rho + r},\ \alpha_2 = \sqrt{\rho - r},\ \tilde{\rho} = \frac{1}{2}(\alpha_1 + \alpha_2),\ \tilde{r} = \frac{1}{2}(\alpha_1 - \alpha_2).$$

Define $\sqrt{\mathbf{z}} = (\tilde{\rho} e^{i\theta/2}, \tilde{r})$.

Enclosing the spectral radius

By a classic result by Collatz we have the following:

Theorem

Let A be an $n \times n$ matrix with non-negative coefficients and let v be an arbitrary positive n-dimensional vector. Set $w = Av$. Then

$$\min_{1 \leqslant i \leqslant n} \frac{w_i}{v_i} \leqslant \rho(A) \leqslant \max_{1 \leqslant i \leqslant n} \frac{w_i}{v_i}. \tag{3}$$

Definitions and Main Result

Partitioned holomorphic dynamical system

A partitioned holomorphic dynamical system $\mathcal{F}$ on $\hat{\mathbb{C}}$ is a finite collection of holomorphic maps

$$f_j \colon U_j \to \hat{\mathbb{C}}, \qquad \text{where } j = 1, \ldots, m, \tag{4}$$

such that

1. the domains U_j are pairwise disjoint open sets in $\hat{\mathbb{C}}$, and each f_j is a proper branched (or unbranched) covering map of U_j onto its image;
2. (Markov property) for every pair of indices i, j, the image $f_i(U_i)$ either contains U_j, or is disjoint from it;
3. (transitivity) for each domain U_j, there exists a finite sequence of maps $f_j = f_{i_0}, f_{i_1}, \ldots, f_{i_k}$ from $\mathcal{F}$, such that the composition $f_{i_k} \circ \cdots \circ f_{i_1} \circ f_{i_0}$ is defined on some open subset $U \subset U_j$ and maps U surjectively onto the union $\cup_{i=1}^m U_i$.

McMullen's matrices

Let $\mathcal{F} = \{(f_j, U_j)\}$ be a partitioned holomorphic dynamical system. Let d_j be the topological degree of f_j, $j = 1, \dots, m$. For $t \geq 0$ introduce the matrix $M(t) = M(\mathcal{F}, t)$ whose (i,j)-th element $m_{ij} = m_{\mathcal{F},t}(i,j)$ is defined as

$$m_{ij} = \begin{cases} d_i \left(\sup_{z \in U_i \cap f_i^{-1}(U_j)} |f_i'(z)| \right)^{-t} & \text{if } U_i \cap f_i^{-1}(U_j) \neq \varnothing \\ 0, & \text{otherwise.} \end{cases}$$

The Main Result

For a square matrix A, let $\varrho(A)$ denote the spectral radius of A (*i.e.* $\varrho(A)$ is the maximum of the absolute values of the eigenvalues of A). For a partitioned holomorphic dynamical system $\mathcal{F}$ define:

$$\delta_{\mathcal{F}} = \inf\{t \geq 0\colon \varrho(M(\mathcal{F}, t)) = 1\}. \tag{5}$$

Let $P(\mathcal{F}, t)$ be the pressure function of $\mathcal{F}$ and $t_{\mathcal{F}} = \inf\{t : P(\mathcal{F}, t) \leqslant 0\}$.

Theorem

For $\mathcal{F}$ such that $\mathcal{F}(\mathcal{U}) \setminus PC(\mathcal{F}) \neq \varnothing$ the number $\delta_{\mathcal{F}}$ is well defined, and $\delta_{\mathcal{F}} \leq t_{\mathcal{F}}$.

Improving the estimates

If $\hat{\mathcal{F}}\colon \hat{\mathcal{U}} \to \hat{\mathcal{F}}(\hat{\mathcal{U}})$ is a refinement of $\mathcal{F}$, then for any $z \in \hat{\mathcal{U}} \setminus PC(\hat{\mathcal{F}})$ and $t > 0$, we have

$$\varrho(M(\mathcal{F}, t)) \leq \varrho(M(\hat{\mathcal{F}}, t)),$$

since the suprema in the definition of M for a refinement $\hat{\mathcal{F}}$ are taken over smaller domains compared to the case of $\mathcal{F}$. In particular,

$$\delta_{\mathcal{F}} \leq \delta_{\hat{\mathcal{F}}}.$$

Convergence of the approximations

Let $\mathcal{F}_k$ be a sequence of partitioned holomorphic dynamical systems such that $\mathcal{F}_{k+1}$ refines $\mathcal{F}_k$ for every $k \in \mathbb{N}$. Then for any $z \in \hat{\mathcal{U}} \setminus PC(\hat{\mathcal{F}})$ and $t > 0$ the sequence $\varrho(M(z, \mathcal{F}_k, t))$ is increasing and therefore converges. Assume that $\mathcal{F}_1$ (and hence, all other $\mathcal{F}_k$) is a restriction of a rational function f and the topological degree of each $\mathcal{F}_k \colon \mathcal{U}_k \to \mathcal{F}_k(\mathcal{U}_k)$ coincides with $\deg f$. Assume that the maximum of the diameters of all tiles of $\mathcal{F}_k$, converges to zero when $k \to \infty$. Recently, F. Przytycki showed that under the above conditions, $\lim \log(\varrho(M(z, \mathcal{F}_k, t))) = P(t, f)$ the pressure of the rational function f. We obtain that $\delta_{\mathcal{F}_k} \to t_f = \mathrm{dim}_{\mathrm{hyp}}(J(f))$, as $k \to \infty$.

Thank you!