More Cantorvals

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Introduction

Let $\sum a_n$ be a convergent series of real numbers. The *achievement set* (or the *set of subsums*) is defined by

$$E(a_n) \ := \ \left\{ x \in \mathbb{R} : \ \exists\, K \in \mathbb{N} \qquad x = \sum_{n \in K} a_n \right\}.$$

A set $B \subset \mathbb{R}$ is *achievable* if there is a series $\sum a_n$ such that $B = E(a_n)$. The n-th remainder

$$r_n := \sum_{i > n} a_i.$$

In particular, r_0 is the sum of the whole series.

S. Kakeya, On the partial sums of an infinite series, Tôhoku Sci. Rep.3(1914) 159-164

Introduction

The Gutrie-Nymann Classification Theorem

The achievement set of any absolutely convergent series is of one of the following four types:

(i) *a finite set;*

(ii) *the union of a finite family of closed and bounded intervals (a multi-interval set);*

(iii) *a Cantor set;*

(iv) *a Cantorval.*

J.A. Guthrie, J.E. Nymann, *The topological structure of the set of subsums of an infinite series*, Colloq. Math. 55(1988) 323-327

J.E. Nymann, R.A.Sáenz, *On the paper of Guthrie and Nymann on subsums of an infinite series*, Colloq. Math. 83(2000) 1-4

Cantorvals

The Guthrie-Nyman set GN.

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Definition

A set $A \subset \mathbb{R}$ is said to be a Cantorval if it is homeomorphic to the Guthrie-Nymann set GN.

Theorem

A nonempty bounded perfect set $P \subset \mathbb{R}$ is a Cantorval if and only if all endpoints of P-gaps are limits of sequences of P-intervals and limits of sequences of P-gaps.

Mendes, P., Oliveira, F., *On the topological structure of the arithmetic sum of two Cantor sets*, Nonlinearity 7(1994) 329-343

Cantorvals

Theorem

A nonempty bounded perfect set $P \subset \mathbb{R}$ is a Cantorval if and only if P-gaps and P-intervals have no common endpoints and the union of all P-intervals is dense in P.

A. Bartoszewicz, M. Filipczak, F. Prus-Wiśniowski, *Topological and algebraic aspects of subsums of series*, Traditional and present-day topics in real analysis, Łódź, 2013 (pp. 345–366)

Cantorvals

Theorem

A bounded set $A \subset \mathbb{R}$ is a Cantorval if and only if it is regularly closed and its boundary $\operatorname{Fr} A$ is a Cantor set.

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J. Marchwicki, P. Nowakowski, F. Prus-Wiśniowski, *Algebraic sums of achievable sets involving Cantorvals*, arXiv: 2309.01589v1[mathCA], 4Sep2023

Cantorvals

Corollary

Let A be an achievement set of an absolutely convergent series. Then:

(i) *A is a finite set if and only if A is nowhere dense and $\operatorname{Fr} A$ is finite;*

(ii) *A is a Cantor set if and only if A is nowhere dense and $\operatorname{Fr} A$ is a Cantor set;*

(iii) *A is a multi-interval set if and only if A is regularly closed and $\operatorname{Fr} A$ is finite;*

(iv) *A is a Cantorval if and only if A is regularly closed and $\operatorname{Fr} A$ is a Cantor set.*

Cantorvals

Until very recently, all known conditions on the series $\sum a_n$, that were sufficient for $E(a_n)$ to be a Cantorval, were applicable only to special multigeometric series.

Ya. Vinnishin et al., *Positive series whose subsum sets are Cantorvals*, Proc. Intern. Geom. Center 12(2)(2019) 26-442 (in Ukrainian)

P. Nowakowski. T. Filipczak, *Conditions for the difference set of a central Cantor set to be Cantorval*, Results Math. (2023) 78:166

M. Pratsiovytyi, D. Karvatskyi, *Cantorvals as sets of subsums for a series connected with trigonometric functions*, arXiv: 2308.13521v1[mathNT], 25Aug2023

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Proposition 2.6

Let $\sum a_n$ be a convergent series of non-negative and non-increasing terms. Then the following conditions are equivalent:

(i) *the achievement set $E(a_n)$ contains an interval;*

(ii) $\lim_{n\to\infty} \Delta_{r_n} F_n > 0$.

Cantorvals

$$F_n(a_k) \; := \; \left\{ x \in \mathbb{R} : \; \exists A \subset \{1, 2, \ldots, n\} \qquad x \; = \sum_{k \in A} a_k \right\}$$

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P_i – connectivity components

$$\|W\| \; := \; \max_{1 \leqslant i \leqslant n} |P_i|$$

Lemma

Let (W_n) be a descending sequence of multi-interval sets. Then $\bigcap_n W_n$ contains an interval if and only if $\lim_n \|W_n\| > 0$.

Definition

F - a finite subset of $\mathbb{R}$, ϵ - a positive number. A set $S \subset F$ is said to be ϵ-close if the distance of all consecutive elements of S does not exceed ϵ.

$$\Delta_\epsilon F \;:=\; \max_{S \subset F} (\max S - \min S).$$

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(ii) $\lim_{n\to\infty} \Delta_{r_n} F_n > 0$.

(iii) $\lim_{k\to\infty} \Delta_{r_{n_k}} F_{n_k} > 0$ *for some increasing sequence (n_k) of indices.*

Cantorvals

the Ferens Cantorvals: $E(m+k-1,\ m+k-2,\ \dots,\ m+1,\ m;q)$ where $m,k \in \mathbb{N},\ k > m$

$$s := \sum_{i=1}^{k-1}(m+i)$$

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Theorem

If $\frac{1}{s-m+1} \leqslant q < \frac{m}{s}$, then $E(m+k-1,\ m+k-2,\ \ldots,\ m+1,\ m; q)$ is a Cantorval. Moreover, its Lebesgue measure equals to the sum of lengths of all E-intervals.

Banakiewicz, M. , Prus-Wiśniowski, F., *M-Cantorvals of Ferens type*, Math. Slovaca 67(4)(2017), 1-12

$m = (m_n)$ – a sequence of postitive integers $\geqslant 2$
$k = (k_n)$ – a sequence of positive integers with $k_n > m_n$
$q = (q_n)$ – a sequence of positive numbers

GF series

$m = (m_n)$ – a sequence of postitive integers $\geqslant 2$
$k = (k_n)$ – a sequence of positive integers with $k_n > m_n$
$q = (q_n)$ – a sequence of positive numbers

the n-th group of terms of the generalized Ferens series $\sum_{i=1}^{\infty} a_i(k, m, q)$:

$$\cdots + (m_n + k_n - 1)q_n + (m_n + k_n - 2)q_n + \ldots + (m_n + 1)q_n + m_n q_n + \cdots$$

$$s_n := \sum_{i=1}^{k_n - 1} (m_n + i)$$

GF Cantorvals

Theorem

If $\sum a_n(m,k,q)$ *is a convergent GF series satisfying the conditions*

$$\forall n \in \mathbb{N} \qquad q_n \leqslant (s_{n+1} - m_{n+1} + 1)q_{n+1} \qquad (\mathrm{GF}_1)$$

and

$$\forall n \in \mathbb{N} \qquad m_n q_n > \sum_{i>n} (s_i + m_i) q_i, \qquad (\mathrm{GF}_2)$$

and then $E(a_n)$ *is a Cantorval.*

GF Cantorvals

Theorem

If $\sum a_n(m, k, q)$ *is a convergent GF series satisfying the conditions* (GF_1) *and* (GF_2)*, then the Lebesgue measure of the Cantorval* $E(a_n)$ *equals to the measure of its interior.*

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Lemma

For every sequence $m = (m_n)$ *of positive integers greater than 1 and for any sequence* (c_n) *with* $c_n > 1$ *and* $\alpha := \sup_n \frac{c_n}{m_n} < 1$, *there are sequences* $k = (k_n) \in \mathbb{N}^{\mathbb{N}}$ *with* $k_n > m_n$ *and* $q = (q_n) \in (0,1)^{\mathbb{N}}$ *such that the GF series* $\sum a_n(m,k,q)$ *is convergent,* $E(a_n)$ *is a Cantorval, and*

$$(s_{n+1} + m_{n+1})q_{n+1} < c_n q_n \qquad \text{for all } n \in \mathbb{N} \tag{1}$$

and

$$c_n < (1-\alpha)(s_n + m_n) \qquad \text{for all } n \in \mathbb{N}. \tag{2}$$

Theorem

There is an achievable Cantor set such that the algebraic sum of any finite number of copies of it remains a Cantor set.

J. E. Nymann, *Linear combinations of Cantor sets*, Colloq. Math. 68 (1995), 259-264

Algebraic sums of achievable sets

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Algebraic sums of achievable sets

Theorem

Let $m, p \in \mathbb{N} \cup \{\infty\}$ be such that $2 \leqslant m \leqslant p$. There is an achievable Cantor set C such that the algebraic sum of k copies of C is

- *a Cantor set for every $k < m$;*
- *a Cantorval for every k such that $m \leqslant k < p$;*
- *an interval for all $k \geqslant p$.*

J. Marchwicki, P. Nowakowski, F. Prus-Wiśniowski, *Algebraic sums of achievable sets involving Cantorvals*, arXiv: 2309.01589v1[mathCA], 4Sep2023

Algebraic sums of achievable sets

Theorem

Every central Cantor set is the algebraic sum of two central Cantor sets of Lebesgue measure zero.

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Theorem

Every infinite achievable set is the algebraic sum of two achievable Cantor sets of Lebesgue measure zero.

We restrict our attention to convergent infinite series $\sum a_n$ of positive and non-increasing terms.
(direct) n-th Kakeya condition : $a_n > r_n$
reversed n-th Kakeya condition: $a_n \leqslant r_n$

$$K(a_n) := \{ n \in \mathbb{N} : a_n > r_n \}, \qquad K^c(a_n) := \{ n \in \mathbb{N} : a_n \leqslant r_n \}$$

Kakeya conditions

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All subsets of $\mathbb{N}$ fall into exactly one of the following three categories: 1) finite subsets, 2) subsets with finite complement, 3) infinite subsets with infinite complement. The first two of those categories showed up among the oldest results on achievement.

Kakeya conditions

Theorem

card $K(a_n) < \infty$ *if and only if* $E(a_n)$ *is a multi-interval set.*

Theorem

If card $K^c(a_n) < \infty$, *then* $E(a_n)$ *is a Cantor set.*

Kakeya conditions

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If card $K^c(a_n) < \infty$, *then* $E(a_n)$ *is a Cantor set.*

The implication in the above theorem can not be reversed which can be seen from the Cantor set $E(1, 1; \frac{1}{4})$, since then $K(a_n) = 2\mathbb{N}$. Thanks to the Guthrie-Nymann Classification Theorem, the above facts imply the following simple corollary.

Corollary

If $E(a_n)$ *is a Cantorval, then card* $K(a_n) =$ *card* $K^c(a_n) = \infty$.

Kakeya conditions

Theorem

For any $K \subset \mathbb{N}$ *with* $\operatorname{card} K = \operatorname{card} K^c = \infty$, *there is a series* $\sum a_n$ *such that* $K(a_n) = K$ *and* $E(a_n)$ *is a Cantor set.*

J. Marchwicki, P. Miska, *On Kakeya conditions for achievement sets*, Results Math.76(2021), article no. 181

Kakeya conditions

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For any $K \subset \mathbb{N}$ *with* $\operatorname{card} K = \operatorname{card} K^c = \infty$, *there is a series* $\sum a_n$ *such that* $K(a_n) = K$ *and* $E(a_n)$ *is a Cantor set.*

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Open Problem

Does for any $K \subset \mathbb{N}$ *with* $\operatorname{card} K = \operatorname{card} K^c = \infty$ *exist a series* $\sum a_n$ *such that* $K(a_n) = K$ *and* $E(a_n)$ *is a Cantorval ?*

In the case of the positive answer, it would prove that in most cases there is no relationship between the set $K(a_n)$ and the topological type of $E(a_n)$. Thus, the Kakeya conditions would give any insight into the topological nature of $E(a_n)$ only in the cases covered by the classic Kakeya's theorems.

Kakeya conditions

The problem is not easy by all means. In all known examples of achievable Cantorvals, mainly generated by multigeometric sequences, al least half of the Kakeya conditions are reversed. Thus, Marchwicki and Miska investigated the auxilliary question of if it is possible to obtain a Cantorval from a sequence with relatively fewer reversed Kakeya conditions and obtained the following result.

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Theorem

For any $0 \leqslant \alpha \leqslant \beta < 1$ there is $\sum a_n$ such that $E(a_n)$ is a Cantorval and $\underline{d}K(a_n) = \alpha$ and $\overline{d}K(a_n) = \beta$.

Additionally, they asked directly if it is possible to construct a series $\sum a_n$ such that $E(a_n)$ is a Cantorval and $dK(a_n) = 1$ (that is, $dK^c(a_n) = 0$) ?

Kakeya conditions

Theorem

For every sequence (m_n) of positive integers convergent to ∞ there is an achievable Cantorval $E(a_n)$ such that

$$\lim_{n\to\infty} \frac{card\{i \leqslant n : \; a_i \leqslant r_i\}}{m_n} = 0.$$

Kakeya conditions

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For every sequence (m_n) of positive integers convergent to ∞ there is an achievable Cantorval $E(a_n)$ such that

$$\lim_{n\to\infty} \frac{card\{i \leqslant n: \ a_i \leqslant r_i\}}{m_n} = 0.$$

In particular, choosing $m_n := n$, we get the classic asymptotic density and the positive answer to the Problem 5.2 from the paper: J. Marchwicki, P. Miska, *On Kakeya conditions for achievement sets*, Results Math. 76(2021), article no. 181 .

Corollary

There is an achievable Cantorval $E(a_n)$ such that the set $\{n: \ a_n \leqslant r_n\}$ has asymptotic density 0.