# Description of all limit distributions of some Markov chains with memory 2

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A Markov chain with memory 2 is a process satisfying

$$Pr(X_{n+1} = i_{n+1}|X_n = i_n, \cdots, X_1 = s_1, X_0 = s_0) = Pr(X_{n+1} = s_{n+1}|X_n = s_n, X_{n-1} = s_{n-1}) \qquad (1)$$

For simplicity, identify the states as $S = \{1, 2, \cdots, r\}$ and assume that the chain is time homogeneous. Then a transition probability matrix $\Pi = (p_{ij})$ defined by

$$p_{ij} := Pr(X_{n+1} = j|X_n = i) \qquad (2)$$

is independent of $n$ and is row stochastic.
As shown in [1] , assuming again time homogeneity, a Markov chain with memory 2 can be conveniently represented via the order -3 tensor $\Pi^{(3)} = (p_{i_1,i_2,i_3})$ defined by

$$p_{i_1,i_2|i_3} := Pr(X_{n+1} = i_3|X_n = i_2, X_{n-1} = i_1) \qquad (3)$$

As noted in [2], the theory of finite Markov chains with positive transition probabilities can be embedded into the theory of limit Gibbs distributions as a trivial particular case, and the Hamiltonians can be considered a natural generalization of the transition probabilities or, more exactly, of their logarithms.
Let $\Omega = S^{Z_+}$ be the sample space of a finite Markov chain with $r$ states. That is, the range space $S$ consists of $r$ elements and $P$ is a measure on $\Omega$ corresponding to a homogeneous Markov chain with a stationary transition matrix $\Pi = (p_{ij})$ and with stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_r)$. We consider the case when all $p_{ij} > 0$.
For arbitrary $\Lambda \subset Z_+$ a configuration $\sigma$ on $\Lambda$ is defined as $\sigma(\Lambda) : \Lambda \to S$. For $\Lambda_{km} = (k, k+1, \cdots, k+m)$, the probability of an arbitrary configuration $\sigma(\Lambda_{km}) = (\sigma(k), \sigma(k+1), \cdots, \sigma(k+m))$ is equal to

$$\pi_{\sigma(k)} \cdot p_{\sigma(k)\sigma(k+1)} \cdot p_{\sigma(k+1)\sigma(k+2)} \cdots p_{\sigma(k+m-1)\sigma(k+m)}$$

$$= \exp\{\ln \pi_{\sigma(k)} + \sum_{i=k}^{k+m-1} \ln p_{\sigma(i)\sigma(i+1)}\}$$

Introducing the Hamiltonian

$$H(\sigma) = -\sum_{i=0}^{\infty} \ln p_{\sigma(i)\sigma(i+1)}, \qquad (4)$$

we see that the interaction energy $\mathcal{G}(\sigma(\Lambda))$ differs from zero only if $\Lambda = \Lambda_{i,1}$ and then $\mathcal{G}(\sigma(\Lambda_{i,1})) = -\ln p_{\sigma(i)\sigma(i+1)}$.
Here $(i, i+1)$ are nearest-neighbours, and in Hamiltonian (3) only interactions between neighbouring spins are taken into account. We consider one more interaction , namely interactions of second neighbours $> i, i+2 <$ .
Let $\Pi^2 = (p_{ij}^{(2)})$ and model specified by the following Hamiltonian

$$H(\sigma) = -\sum_{i=0}^{\infty} \ln p_{\sigma(i)\sigma(i+1)} - \sum_{i=0}^{\infty} \ln p^{(2)}_{\sigma(i)\sigma(i+2)}, \qquad (5)$$

Such model is called the model with competing interactions [3]. We describe all limit distributions of some Markov chains with memory 2.

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